Expectations and Frictions: Lessons from a Quantitative Model with Dispersed Information*

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Abstract

What are the macroeconomic implications of informational frictions in a quantitative business cycle model? We develop a general solution method that allows enriching a standard medium-scale DSGE model with dispersed information and estimating using Bayesian techniques with comprehensive macroeconomic and expectation data. We draw important conclusions regarding the role information frictions play in business cycles. First, the degree of informational friction varies significantly across shocks and indicates important departures from complete information. Second, information frictions cannot substitute standard frictions to generate inertia in macroeconomic variables. When disciplining our model with expectational data, standard frictions such as habits and investment costs increase substantially despite the pervasive informational frictions. Third, simulated data from the model can match standard empirical measures of informational frictions when using data on forecast revisions in the estimation. Fourth, we use our model to assess whether those empirical estimates are reliable measures of informational frictions. We find that these measures are more sensitive to information frictions than to standard frictions. Finally, compared with its full-information counterpart, our model generates a weaker recession after inflationary price markup shocks and stronger real effects but a less inflationary response to monetary shocks.

Keywords: Business cycles, DSGE model, informational frictions, higher-order beliefs
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1 Introduction

Modern macroeconomics rests upon the framework of full-information rational expectations (FIRE). Despite the considerable achievements of the rational expectations paradigm, it faces two important limitations. First, it relies on very persistent exogenous shocks and various frictions, such as habit formation in consumption and adjustment cost in investment, to replicate the slow adjustment of aggregate variables. These frictions are often referred to as “ad hoc” (Chari et al. (2009); Angeletos and Huo (2021)), because they lack robust micro-foundation and empirical evidence, particularly for the values typically used in macroeconomic models. Second, FIRE models are inconsistent with the substantial heterogeneity and underreaction in expectations about macroeconomic conditions observed in the survey data (Mankiw et al. (2004); Coibion and Gorodnichenko (2015); Coibion et al. (2018)).

On another front, an emerging body of literature, pioneered by Mankiw and Reis (2002), Woodford (2002) and Sims (2003), argues that informational frictions can address the aforementioned shortcomings in FIRE models. This is because the gradual learning process induced by informational imperfections generates a more realistic expectation formation process and *qualitatively* reproduces the slow response of aggregates to shocks observed in the data.

Despite the promising theoretical and empirical results suggesting the role of informational frictions in replicating key empirical regularities of the macroeconomic and expectation variables, there is still scarce work evaluating rigorously the quantitative performance of models with information imperfections. This is because existing methods to solve DSGE models featuring informational imperfections still have limited applicability, especially when linking the model to the data. The challenge faced by this strand of the literature lies in the complexity of solving DSGE models when we relax the assumption of full information. Under incomplete information, agents must form an infinite regress of expectations, as first pointed out by Townsend (1983). Then, state representations of DSGE models with such informational structure become infinite-dimensional.

The current stance of the literature leaves several unanswered questions. How does the introduction of information frictions change the estimates for standard frictions in macroeconomic model? What is the best expectation data to use for estimating macroeconomic models with or without informational frictions? Can the model reproduce key empirical measures of departures from FIRE? And how do standard and informational frictions interact to shape these deviations? How does the introduction of informational frictions change our understanding of the drivers of business cycles and the propagation of shocks?

To answer these questions, we augment a medium-scale DSGE model along the lines of Christiano et al. (2005) and Smets and Wouters (2007) with an informational friction in the form of dispersed information about the state of the economy. This form of informational friction is standard in the literature[1] and dates back to Lucas (1972)'s seminal island model. Importantly, it has proven to be consistent with the empirical evidence on the underreaction of average expectations to aggregate shocks.

We develop a novel method to solve and estimate DSGE models with dispersed and exogenous information. The method bridges Uhlig's (2001) undermined coefficient method for full-information models with the solution methods from Nimark (2008) and Melosi (2017) for dispersed information models that truncates hierarchy of beliefs to a finite order. Our method improves upon previous solutions by allowing for the inclusion of endogenous state variables into the system of log-linearized equilibrium conditions, which allows using medium and large-scale quantitative models. The solution is sufficiently fast to perform Bayesian estimation with standard techniques.

This differs from the previous literature on quantitative models with informational frictions, which was limited to the estimation of small-scale DSGE models using solely macroeconomic data or including little information from survey expectation data.

Equipped with this new toolbox for working with this class of models, we estimate the model featuring both standard frictions and incomplete information (the "DI model) and compared it with its commonly used full-information counterpart (the "FI" model). Both models are estimated using Bayesian techniques with two different datasets: one including only U.S. macroeconomic data, as common practice in the literature, and another augmented with expectation data on forecast revisions for real output, consumption, investment, inflation and nominal rate from the U.S. Survey of Professional Forecasters (SPF).

Several interesting results arise from our estimation exercise. When we include expectation data in the DI model estimation, we find that parameter estimates for both models change substantially. Under the complete dataset, the estimated persistence of several shocks declines substantially, in particular for the monetary, investment and preference shocks. Moreover, habit formation and the adjustment cost in investment rise significantly, and the Taylor-rule response to inflation and past interest rates both increase.

This increase in the level of "ad hoc" parameters goes in contrast with what one would expect from the theoretical literature on information frictions, which suggests that incomplete information can be an alternative device to generate the inertia in macroeconomic variables consistent with the data. Our results suggest that both standard and information frictions are important in explaining the macroeconomic and expectation data.

In the DI model, including data on forecast revisions increases both the estimated levels of information frictions and the variances of the shocks. Interestingly, there is a considerable decrease in the variance of measurement errors for expectational variables, a first indication that the DI model provides a better explanation for expectation data, which we confirm further when

[1] See for instance Woodford (2002), Nimark (2008), Angeletos and La'O (2013), Melosi (2017), among others.

comparing the marginal likelihoods of the FI and DI models.

Introducing information frictions also increases the variances of shocks for both datasets, with a more pronounced effect observed in the complete dataset. The higher shock volatility is offset by the impact of information frictions, resulting in less pronounced responses. Overall, the estimates derived from the complete dataset exhibit stronger standard frictions, including habits, adjustment costs, price and wage indexation, and robust informational frictions across all shocks. Monetary shocks display comparatively smaller information frictions, although they remain distant from the full information benchmark.

In our view, the estimation exercise highlights two important lessons. First, our framework suggests that both standard and informational frictions are essential to fit the macroeconomic and expectation data. Second, parameter estimates, even for the model with complete information, are not robust to the inclusion of expectation data.

We then investigate if the estimated DI model is able to replicate unmatched moments of the expectation data. Specifically, using the approach by Coibion and Gorodnichenko (2015) (CG henceforth), we estimate empirical Kalman gains, which measure the degree to which agents incorporate new information into their forecasts. These estimates represent the bite of information frictions. In general, Kalman gain estimates from the model are very close to their empirical counterparts.

We also evaluate what are the more informative variables associated with the expectation to use to discipline macroeconomic models with incomplete information. Our model suggests that there strong link between forecast revisions and informational frictions. This is the main reason why we use data on forecast revisions. However, the few papers that use expectation data usually use data on forecasts (Del Negro and Eusepi; 2011; Del Negro et al.; 2015; Melosi; 2017).

We confirm this premise by re-estimating the DI model using two alternative datasets: i) one including data on forecasts in place of revision forecasts, and ii) another including data on forecast errors. Then, we evaluate which model can replicate best the empirical Kalman gains. Our findings suggest that overall, the model using data on forecast revisions has estimates that are closer to the empirical evidence.

We also shed some light on the reliability of empirical Kalman gains as measures of informational frictions. As discussed by Angeletos and Huo (2021) and Angeletos et al. (2020), Kalman gains are determined by the interaction of informational frictions with the other frictions in the model.

In our model, this is even more prominent as we have various sources of shocks and frictions. To quantitatively evaluate this point, we vary the level of each friction to see which of them is most important to explain a particular empirical Kalman gain. In this sense, our exercise extends the analytical results in Angeletos and Huo (2021).

We have two important findings that reinforce the relevance of such measures. First, empirical

Kalman gains are remarkably stable when varying real and nominal frictions, with some notable and interesting exceptions. Consumption Kalman gain is highly sensitive to habits, inflation Kalman gain is sensitive to price indexation and nominal rate Kalman gain is sensitive to monetary rule smoothing.

Second, empirical Kalman gains are very sensitive to information friction parameters. Interestingly, information friction for each shock does not affect all empirical Kalman gains computed from the model. For instance, the information friction on the preference shock plays a major role for Kalman gains of consumption and output only. Also intuitively, the Kalman gain of interest is mostly driven by imperfect information about monetary policy shocks.

Our last contribution is to study what are the implications of the estimated model with dispersed information to our understanding of the drivers of business cycles and the propagation of aggregate shocks. Performing a forecast error variance decomposition (FEVD) exercise, we find that investment shocks have a more prominent role in explaining output in the DI compared to its full-information counterpart, in a similar vein with the results by Auclert et al. (2020), whose departure from FIRE in a heterogenous-agent New Keynesian framework.

Finally, we compare impulse response functions (IRFs) to shocks of the DI and FI models. The baseline FI and DI models generate similar impulse responses of macroeconomic variables, since they are both estimated to reproduce these moments of the data. However, only the DI model can reproduce the slow adjustment of expectations to shocks. Notably, the introduction of information frictions is key to reproduce the impulse responses to an investment shock. Those IRFs exhibit a higher response on impact and lower persistence when informational frictions are turned off, showing that dispersed information is key to dampen the impact of the shock and generate a slow adjustment of macroeconomic aggregates to it.

Outline. The remainder of this paper is organized as follows. Section 1.1 reviews the related literature. Section 2 presents the details of the model and its equilibrium relations. Section 3 explains the new solution method we developed. Section 4 details the dataset used, the estimation procedure, and its results. Section 5. Section 6 performs different quantitative exercises to evaluate the ability of the model with DI but without some of the reduced-form frictions to explain the data compared to the standard full-information model with all frictions. Section 7 concludes.

1.1 Related literature

This paper is related to several strands of the macroeconomic literature.

First, our results compare directly with those of the extensive literature on quantitative business cycle models, built upon the full-information paradigm, in which Christiano et al. (2005) and Smets and Wouters (2007) are the most prominent examples. Specifically, the only crucial difference between our model and the ones in the aforementioned papers is the informational structure. Moreover, the key exercise of our paper is to confront the fit of a model featuring DI but without some of the reduced-form frictions with the mainstream full-information model. We also provide an assessment of which expectation data to use when accounting for informational frictions.

Second, our paper relates to a theoretical literature that suggests that informational frictions can be an alternative - and with better microfoundations - a device to generate inertia in macroeconomic models. Our model lies within the class of noisy information starting with Woodford (2002) both learning about fundamentals and even stronger sluggishly adjustment of higher-order expectations. Angeletos and La'O (2010) and Angeletos and La'O (2013) embed a calibrated RBC model with this information setup to explain business cycles. However, they do not include a full set of other competing frictions to explain the inertia in the data.

Our paper integrates the growing but still small quantitative literature estimating incomplete information models. Melosi (2014) estimates a NK model in which firms have DI about the money growth and technological shocks. The author shows that the model with DI fits the data better than a model with sticky prices a la Calvo (1983) with indexation. Also, Melosi (2017) estimates a Calvo-like sticky price model with DI about monetary, technological, demand shocks. He finds that there is a signalling channel of monetary policy in the US and that the model fits better the data than a New Keynesian model with habits in consumption and indexation.

Third, our paper relates to alternative methods for solving dispersed information models in general settings. Our setting extends methods from Nimark (2008) and Melosi (2017) that truncates the hierarchy of beliefs to a finite order to avoid the infinite regress of higher-order expectations. There are other alternatives using frequency domain techniques. The more prominent alternative is due to Huo and Takayama (2023) that use a combination of the Wiener-Hopf prediction formula and the Kalman filter that allows a tractable finite-state representation for the equilibrium. They apply their model to a small dynamic beauty context model and a HANK-type model with incomplete information.

Huo and Takayama (2022) uses the same method to solve an RBC model with TFP and confidence shocks as in Angeletos and La'O (2013). Their model does not include standard nominal and 'ad hoc' frictions but does a good job of matching main business cycle moments and conditional responses identified in empirical VAR.

While our solution method relies on an approximation, we solve a medium-scale DSGE model fast enough to pursue a full Bayesian estimation. As shown by Nimark (2017), this approximate solution is accurate for the class of models that the importance of the order of expectation decreases with the order, which our model satisfies.

Relatedly, Angeletos et al. (2018) adopt a different approach to solve a DSGE with imperfect and dispersed information. They propose a model with heterogeneous prior and using the limiting case of vanishing variance of private signals. In that way, they can solve and estimate the model by shutting down the learning process and representing the expectations hierarchy into a low-dimensional state variable. The advantage of this method is that it has virtually the same computational burden as the full-information model. The drawbacks are that the dispersion in expectations is shut down and apply only to "confidence shocks" that do not change fundamentals, but affect expectations.

Our application evaluates the relevance of the informational frictions, which are shut down in their method. For a fair comparison with the full information benchmark, we do not compare with confidence or noise shocks that provide an additional source of explanation for business cycles.

Chahrour and Ulbricht (2023) uses a business cycle accounting approach for quantifying the importance of information as a source of business cycle fluctuations regardless informational structure and provide conditions for confidence shocks playing a major role.

We also relate to a quantitative literature signal extraction into DSGE models with common information. Lippi and Neri (2007) estimate a small-scale DSGE model with imperfect information, discretionary monetary policy and indicator variables with potential information role for stabilization policy. Collard et al. (2009) estimate a standard New Keynesian (NK) model under three alternative forms of imperfect information: confusion between temporary and persistent shocks, unobserved variation in the inflation target of the central bank, and persistent mis-perceptions about the state of the economy. The authors argue that the inclusion of imperfect information improves the fit of the traditional NK model according to the marginal likelihood criterion. Neri and Ropele (2011) contribute to this literature by estimating a NK model with imperfect information for the euro area using real-time data. Thus, they depart from the prior works mentioned above that use only *ex-post* revised macroeconomic data.

Finally, our paper also is related the literature on evidence of information frictiosn in surveys on expectation Coibion and Gorodnichenko (2012, 2015); Bordalo et al. (2020); Angeletos et al. (2020). We contribute to this literature in two aspects. First, we provide an similar empirical but more direct strategy to estimate the degree of informational frictions from data. Second, and more importantly, we evaluate whether the measure proposed by CG's empirical strategy can really uncover the informational frictions.

2 Model

The model is a standard medium-scale DSGE model along the lines of Christiano et al. (2005) and Smets and Wouters (2007). It features sticky prices and wages, variable capital utilization, fixed cost in intermediate production, and five frictions that generate endogenous persistence in the model: (i) habit formation in consumption, (ii) adjustment cost in investment, (iii, iv) partial indexation of prices and wages to past inflation, and (v) smoothing in the monetary policy rule. In the baseline specification of the DI model, we shut off three frictions that induce inertia in the model, namely: habit persistence in consumption and indexation of prices and wages to lagged inflation. We discuss this choice in the results section. That said, in the following description of the model, we derive it with all the frictions, including the informational imperfection, and later we detail the frictions that will be turned off.

The stochastic dynamics is driven by seven orthogonal exogenous shocks. The model includes total factor and investment-specific productivity shocks, a preference shock, wage and price markup shocks, and government expenditure and monetary policy shocks.

The innovation of our work is the introduction of dispersed information (DI) to this environment. Specifically, households and firms do not observe perfectly the shocks hitting the economy. Instead, they receive noisy idiosyncratic signals about them.

Timing

Time is discrete and each period contains two stages. In the first stage, shocks and signals are realized, intermediate goods firms choose optimal prices, and households choose consumption, investment, installed capital, and its utilization level, and set optimal wages, based on information from their signals. In the second stage, rental rates and wages of differentiated labor are uncovered. Competitive final good firms buy intermediate goods to sell the final good to households. Competitive labor packers use the supply of differentiated labor from households and sell a homogeneous labor bundle to intermediate firms. Intermediate firms rent capital and hire labor to produce the intermediate goods.

This timing protocol is standard in the literature[2] and ensures two features. First, all markets clear. Competitive final good firms ensure that the supply of final good matches consumption and investment demands. Given prices of intermediate goods, differentiated wages, and rental rate chosen at stage 1, firms allocate capital and labor to accommodate the demand from final good firms. Finally, given the wages set at the first stage, labor packers aggregate the differentiated labor services from households and supply homogeneous labor to intermediate firms such that the labor market clears.

Second, intermediate good firms do not use information from the production process to extract information from aggregate variables. This implies that both intermediate firms and households use only information from their signals to form expectations.

[2]See for instance Nimark (2008), Angeletos and La'O (2011).

Final good firms

The homogeneous final good Y_t is a bundle of intermediate goods, $Y_{i,t}$, where the index $i \in [0,1]$ denotes the continuum of intermediate firms. The production function is given by

$$Y_t = \left(\int_0^1 (Y_{i,t})^{\frac{1}{1+\mu_t^p}} \, di \right)^{1+\mu_t^p}, \tag{1}$$

where μ_t^p is the time-varying price mark-up of intermediate goods following the process $log(1+\mu_t^p) = log(1+\bar{\mu}^p) + x_t^p$, and $\bar{\mu}^p$ is the steady-state value of the price mark-up. The price mark-up shock x_t^p follows

$$x_t^p = \rho_p x_{t-1}^p + \varepsilon_t^p, \quad \varepsilon_t^p \sim \mathcal{N}(0, \sigma_p^2). \tag{2}$$

Final goods firms operate in a perfectly competitive market and sell the final goods to households for a price P_t. They solve a usual profit maximization problem

$$P_t Y_t - \int_0^1 P_{i,t} Y_{i,t} di \tag{3}$$

subject to (1). From the first-order conditions and the zero-profit condition, we derive the demand for intermediate production

$$Y_{i,t} = \left(\frac{P_{i,t}}{P_t} \right)^{-\frac{1+\mu_t^p}{\mu_t^p}} Y_t, \tag{4}$$

where $P_t = \left(\int_0^1 (P_{i,t})^{-\frac{1}{\mu_t^p}} \, di \right)^{-\mu_t^p}$ is the price level.

Intermediate good firms

Each intermediate producer is a monopolistic competitive firm $i \in [0,1]$ that uses labor and capital to produce their goods. Good i is produce using technology

$$Y_{i,t} = e^{x_t^a} (K_{i,t})^{\alpha} (\gamma^t L_{i,t})^{1-\alpha} - \gamma^t \Phi_p, \tag{5}$$

where γ represents the labor-augmenting deterministic growth rate of productivity and x_t^a is a total factor productivity shock. $K_{i,t}$ and $L_{i,t}$ denote the amount of capital and labor demanded by firm i at period t respectively, and Φ_p is a fixed cost included in the production function.[3] The

[3] The fixed cost must be scaled by γ^t. Otherwise, it cost would become increasingly smaller in terms of production over time.

productivity shock follows

$$x_t^a = \rho_a x_{t-1}^a + \varepsilon_t^a, \quad \varepsilon_t^a \sim \mathcal{N}(0, \sigma_a^2). \tag{6}$$

Intermediate firms are subject to a Calvo (1983)-like pricing friction: in every period, only a fraction $1 - \xi_p$ of firms can adjust prices. Firms not allowed to reset prices apply the following indexation rule

$$P_{i,t} = (\Pi_{t-1})^{\iota_p} \left(\bar{\Pi}\right)^{1-\iota_p} P_{i,t-1}, \tag{7}$$

where $\Pi_t \equiv P_t/P_{t-1}$ is gross inflation and $\bar{\Pi}$ is its steady-state value. Firms that are able to reoptimize prices choose the level $P_{i,t}^*$ that maximizes their expected discounted flow of profits given by

$$E_{it} \sum_{s=0}^{\infty} (\beta\xi_p)^s \Lambda_{t,t+s} \left[\left(X_{t,t+s} P_{i,t}^* - MC_{i,t+s}\right) Y_{i,t+s}\right], \tag{8}$$

subject to $Y_{i,t+s} = \left(\frac{X_{t,t+s}P_{i,t}^*}{P_{t+s}}\right)^{-\frac{1+\mu_{t+s}^p}{\mu_{t+s}^p}} Y_{t+s}$. $MC_{i,t}$ is the marginal cost of firm i at period t. $\Lambda_{t,t+s}$ is the household's stochastic discount factor between periods t and $t+s$ and $X_{t,t+s}$ is the indexation between the same periods. Consistently with the indexation rule (7), $X_{t,t+s}$ is given by

$$X_{t,t+s} = \begin{cases} \bar{\Pi}^{(1-\iota_p)s} \prod_{j=1}^{s} \left(\Pi_{t+j-1}^{\iota_p}\right) & \text{if} \quad s \geq 1 \\ 1 & \text{if} \quad s = 0. \end{cases} \tag{9}$$

Given the optimal prices and the indexation rule, the price level has the following law of motion

$$P_t = \left[\xi_p \left(\Pi_{t-1}^{\iota_p} \bar{\Pi}^{1-\iota_p} P_{t-1}\right)^{-\frac{1}{\mu_t^p}} + (1-\xi_p) \left(\int_0^1 P_{i,t}^* di\right)^{-\frac{1}{\mu_t^p}}\right]^{-\mu_t^p}. \tag{10}$$

Labor packers

There is a continuum of households indexed by $h \in [0, 1]$, each supplying differentiated labor services. Following Erceg et al. (2000), there are labor packers that hire labor from the households and aggregate them according to

$$L_t = \left(\int_0^1 L_{h,t}^{\frac{1}{1+\mu_t^w}} dh\right)^{1+\mu_t^w}, \tag{11}$$

where μ_t^w denote the agency's wage mark-up such that $log(1 + \mu_t^w) = log(1 + \bar{\mu}^w) + x_t^w$, and $\bar{\mu}^w$ is the steady-state value of the wage mark-up. The shock follows the process

$$x_t^w = \rho_w x_{t-1}^w + \varepsilon_t^w, \quad \varepsilon_t^w \sim \mathcal{N}(0, \sigma_w^2). \tag{12}$$

Labor packers pay the wage $W_{h,t}$ for each household h, and sell a homogeneous labor service to intermediate firms at a cost W_t. Agents maximize profits

$$W_t L_t - \int_0^1 W_{h,t} L_{h,t} dh \tag{13}$$

subject to (11). Thus, the labor demand for each household h's labor service is given by

$$L_{h,t} = \left(\frac{W_{h,t}}{W_t}\right)^{-\frac{1+\mu_t^w}{\mu_t^w}} L_t, \tag{14}$$

where $W_t = \left(\int_0^1 (W_{h,t})^{-\frac{1}{\mu_t^w}} dh\right)^{-\mu_t^w}$ is the nominal wage index.

Households

Households $h \in [0,1]$ derive utility from consumption and leisure. In order to maximize their expected utility, they choose consumption ($C_{h,t}$), holdings of government bonds ($B_{h,t}$), installed capital level ($K_{h,t}$) and its utilization rate ($U_{h,t}$). The capital rented to firms, K_t^u, is determined by the installed capital and the utilization rate. The objective function that each household h optimizes is given by

$$U = E_{ht}\left[\sum_{s=0}^{\infty} \beta^s e^{x_{t+s}^c}\left(ln\left(C_{h,t+s} - \varphi C_{h,t+s-1}\right) - \frac{L_{h,t+s}^{1+\chi}}{1+\chi}\right)\right], \tag{15}$$

where $C_{h,t}$ is consumption and $L_{h,t}$ denotes the supply of differentiated labor services of household h at period t. Households' preferences display external habit persistence, captured by the parameter φ, while χ is the inverse of the Frisch elasticity of labor supply. $E_{ht}[\cdot]$ is the expectation operator conditional on households h's information set, and x_t^c is a preference shock that follows

$$x_t^c = \rho_c x_{t-1}^c + \varepsilon_t^c, \quad \varepsilon_t^c \sim \mathcal{N}(0, \sigma_c^2). \tag{16}$$

The capital stock $K_{h,t}$ owned by household h evolves according to

$$K_{h,t} = (1-\delta)K_{h,t-1} + e^{x_t^i}\left(1 - S(I_{h,t}/I_{h,t-1})\right) I_{h,t}, \tag{17}$$

where $S(I_t/I_{t-1})$ is the adjustment investment cost function that denotes the share of investment which does not become new capital. As in Christiano et al. (2005), the cost function $S(\cdot)$ has the following properties: $S(\gamma) = S'(\gamma) = 0$, and $S''(\gamma) = s'' > 0$.

The investment-specific technological shock x_t^i follows

$$x_t^i = \rho_i x_{t-1}^i + \varepsilon_t^i, \quad \varepsilon_t^i \sim \mathcal{N}(0, \sigma_i^2) \tag{18}$$

Households rent to firms an effective amount of capital $K_{h,t}^u$ given by

$$K_{h,t}^u = U_{h,t} K_{h,t-1}, \tag{19}$$

where $U_{h,t}$ is the level of capital utilization. They receive $R^k K_{h,t}^u$ for renting capital but pay a cost $a(U_{h,t})K_{h,t-1}$ in terms of the consumption good. Following Christiano et al. (2005), this function has the properties $a(\bar{U}) = 0$ and $a''(\bar{U}) = a''$, where $\bar{U}$ is the value of the capital utilization rate in the steady-state.

The households' budget constraint is given by

$$\begin{aligned} &P_t C_{h,t} + P_t I_{h,t} + B_{h,t} + P_t a(U_{h,t}) K_{h,t-1} + Q_{t+1,t} A_{h,t} \leq \\ &R_{t-1} B_{h,t-1} + W_{h,t} L_{h,t} + R_t^k U_{h,t} K_{h,t-1} + P_t A_{h,t-1} + T_{h,t}, \end{aligned} \tag{20}$$

in each period. $T_{h,t}$ denote net transfers from the government, $A_{h,t}$ is a vector of one-period state-contingent securities and $Q_{t+1,t}$ is the price of such asset.[4]

Households supply labor in a market with monopolistic competition subject to a wage-setting friction. Following Erceg et al. (2000), in every period only a fraction $1 - \xi_w$ of households are able to optimize their wages. Those households choose their optimal wage by setting a mark-up over the marginal rate of substitution between consumption and labor. Households who are not able to optimize update their wage using a indexation rule given by

$$W_{h,t} = (\Pi_{t-1})^{\iota_w} \left(\bar{\Pi}\right)^{1-\iota_w} \gamma W_{h,t-1}, \tag{21}$$

which is a geometrically weighted average of steady-state wage growth ($\gamma\bar{\Pi}$) and last period wage growth ($\gamma\Pi_{t-1}$).

Each household minimizes their expected discounted labor disutility

$$E_{ht}\left[\sum_{s=0}^{\infty} (\beta\xi_w)^s \left(-\frac{L_{j,t+s}^{1+\chi}}{1+\chi}\right)\right] \tag{22}$$

subject to the budget constraint (20) at all periods $s \in [0, \infty)$ and to the period $t + s$ nominal

[4] The assumption of a complete set of state-contingent securities guarantees that all households h will make the same consumption and saving choices. This is true despite the fact that they have differentiated wages and form expectations based on idiosyncratic signals. This assumption is for tractability. In that way, it is not required to keep track of the stationary distribution of capital as in heterogeneous-agent models. Thus, standard techniques of log-linearization to solve DSGE models can be applied even with informational frictions in the households' decisions.

wage $W_{h,t+s} = X^w_{t,t+s} W^*_{h,t}$ and

$$X^w_{t,t+s} = \begin{cases} (\gamma \bar{\Pi}^{1-\iota_w})^s \prod_{j=1}^{s} \left(\Pi^{\iota_w}_{t+j-1} \right) & \text{if} \quad s \geq 1 \\ 1 & \text{if} \quad s = 0. \end{cases} \tag{23}$$

Given optimal prices and the indexation rule, the aggregate wage level has the following law of motion

$$W_t = \left[\xi_w \left(\gamma \Pi^{\iota_w}_{t-1} \bar{\Pi}^{1-\iota_w} W_{t-1} \right)^{-\frac{1}{\mu^w_t}} + (1-\xi_w) \left(\int_0^1 W^*_{h,t} dh \right)^{-\frac{1}{\mu^w_t}} \right]^{-\mu^w_t} . \tag{24}$$

Government Policies

The central bank sets the nominal interest rate according to a standard Taylor rule with smoothing according to

$$\frac{R_t}{\bar{R}} = \left(\frac{R_{t-1}}{\bar{R}} \right)^{\phi_R} \left[\left(\frac{\Pi_t}{\bar{\Pi}} \right)^{\phi_\pi} \left(\frac{Y_t}{\bar{Y}} \right)^{\phi_y} \right]^{(1-\phi_R)} e^{x^r_t}, \tag{25}$$

where variables with a bar denote the steady-state values and x^r_t is the monetary policy shock, which follows

$$x^r_t = \rho_r x^r_{t-1} + \varepsilon^r_t, \quad \varepsilon^r_t \sim \mathcal{N}(0, \sigma^2_r). \tag{26}$$

For simplicity, I assume that the monetary authority responds to deviations of output to its steady-state value instead of the natural output.

The government budget constraint is the following

$$P_t G_t + R_{t-1} B_{t-1} = B_t + T_t \tag{27}$$

where T_t and B_t are total lump-sum taxes and bonds, respectively. Government expenditure follows a simple stochastic process given by $G/Y = g_y + x^g_t$, where $g_y \equiv \bar{G}/\bar{Y}$ is the steady-state ratio of government expenditure to output and x^g_t is a government expenditure shock with process

$$x^g_t = \rho_g x^g_{t-1} + \varepsilon^g_t, \quad \varepsilon^g_t \sim \mathcal{N}(0, \sigma^2_g). \tag{28}$$

Resource constraint and market clearing

The aggregate resource constraint

$$C_t + I_t + G_t + a(U_t) K_t = Y_t, \tag{29}$$

is derived by integrating households' budget constraint over h, and combining it with the zero profit condition of final goods firms and labor packers and the government budget constraint.

Market clearing in labor and capital markets holds

$$K_t^u = \int_0^1 K_{i,t} di, \quad L_t = \int_0^1 L_{i,t} di.$$

The bond supply and the transfers

$$B_t = \int_0^1 B_{h,t} dh, \quad T_t = \int_0^1 T_{h,t} dh,$$

are consistent with the government spending rule and the public budget constraint (27).

Information and signal extraction

Intermediate good firms and households do not observe perfectly the structural shocks driving the economy, but instead receive noisy idiosyncratic signals about them. Formally, the signal follows the process

$$s_{j,t}^l = x_t^l + v_{j,t}^l, \quad v_{j,t}^l \sim \mathcal{N}(0, \tau_l^2), \tag{30}$$

where $l \in \{a, c, i, g, p, w, r\}$ denote each type of shock and $j \in [0, 1]$ is a index that pools both intermediate good firms i and households h. This assumption implies that firms and household receive signals with the same properties and are subject to the same degree of informational frictions. Hence, an agent j's information set is described as

$$\mathcal{I}_t^j = \{s_{j,\tau}^a, s_{j,\tau}^c, s_{j,\tau}^i, s_{j,\tau}^g, s_{j,\tau}^p, s_{j,\tau}^w, s_{j,\tau}^r : \tau \leq t\}. \tag{31}$$

Given that households and firms have the same unit mass, their average expectation is the same and denoted by $\bar{E}_t[\cdot] \equiv \int_0^1 E_{j,t}[\cdot] dj$, where $E_{j,t}[\cdot] \equiv E[\cdot|\mathcal{I}_t^j]$ denotes the j's individual expectation. Given the AR(1) structure of the shocks and signal structure (30), j's rational expectation about the shock x_t^l is computed by using the Kalman filter such that

$$E_{j,t}[x_t^l] = E_{j,t-1}[x_t^l] + \bar{k}_l \left[s_{j,t}^l - E_{j,t-1}[s_{j,t}^l] \right] \tag{32}$$

where denotes $\bar{k}_l$ the steady-state Kalman gain of shock l given by

$$\bar{k}_l = \frac{\tilde{p}_l}{\tilde{p}_l + r_l^2} \tag{33}$$

where $r_l \equiv \frac{\tau_l}{\sigma_l}$ denotes the noise-to-signal ratio of signal l and $\bar{p}_l$ can be computed by the Riccati equation

$$\tilde{p}_l^2 - \left[1 - (1-\rho_l^2)r_l\right]\tilde{p}_l - r_l^2 = 0.$$

Note that $\tilde{p}_l$ is defined as the ratio of the individual expectation steady-state variance and the shock variance, $\tilde{p}_l = \frac{\bar{p}_l}{\sigma_l^2}$.Thus, the Kalman gain is a function of the shock's parameters, (ρ_l, σ_l), and the signal's noise (τ_l), $\bar{k}_l = k(\rho_l, \sigma_l, \tau_l)$.

The Kalman gain is the key parameter that measures the degree of informational friction, *conditional on the shock's parameters*. In other words, conditional on (ρ_l, σ_l) there is a one-to-one relationship between $\bar{k}_l$ and the signal's noise ratio τ_l. We get back on this when discussing priors choices.

Moreover, the average expectation is given by

$$\bar{E}[x_t^l] = (1-\bar{k}_l)\bar{E}_{t-1}[x_t^l] + \bar{k}_l x_t^l = (1-\bar{k}_l\rho_l)\bar{E}_{t-1}[x_{t-1}^l] + \bar{k}_l x_t^l. \tag{34}$$

Thus, the current expectation is a weighted average between the past expectation and the actual shock, whose weight is given by the Kalman gain.

If $\bar{k}_l = 1$ ($\tau_l \to \infty$) the model boils down to the full information benchmark. If $\bar{k}_l = 0$ ($\tau_l \to 0$), the average expectation is zero (equals to the unconditional expectation)[5].

2.1 Detrending and log-linearized model

Before showing the system of log-linearized equations that characterizes the model, we discuss the detrending procedure.

All real variables grow along with the productivity trend, so they are detrended as follows: $\hat{Z}_t = \frac{Z_t}{\gamma^t}$, for any real variable Z_t. Nominal variables grow along with the price level P_t, hence they are stationarized using the following procedure: $\hat{Z}_t = \frac{Z_t}{P_t}$.[6]

The optimal prices and wages are detrended by dividing them to the price level of last period: $\hat{P}_{i,t}^* = P_{i,t}^*/P_{t-1}$, while $\hat{W}_{h,t}^* = \hat{W}_{h,t}^*/\gamma^t P_{t-1}$.[7]

Stationary variables are transformed by taking their log-deviation to steady-state value as follows: $z_t = log(Z_t/\bar{Z})$, for any stationary variable Z_t. Thus, lower case variables denote log-deviation from steady-state of the upper case variables.

[5]Zero is the solution for the equation $\bar{E}[x_t^l] = (1-\rho_l)\bar{E}_{t-1}[x_{t-1}^l]$

[6]One exception is the Lagrange multiplier of the budget constraint, Λ_t, that must be normalized to $\hat{\Lambda}_t = \Lambda_t\gamma^t P_t$, since it reflects the marginal utility of an additional unit of money, which declines as consumption grows at rate γ and the price level rises.

[7]This simplifies the derivation as $E_{h,t}[\hat{W}_{h,t}^*] = \hat{W}_{h,t}^*$ and $E_{i,t}[\hat{P}_{i,t}^*] = \hat{P}_{i,t}^*$ since observe P_{t-1}. This would not be true if we detrended them using P_t.

For brevity, all log-linearized equations are provided in equation (65) in the Appendix A.2. The derivation of the log-linearized equations above can be found in the Online Appendix.

Here we emphasize two key equations to clarify how dispersed information affects the equilibrium conditions.

Consider the Euler equation given by

$$c_t = \frac{\varphi/\gamma}{1+\varphi/\gamma} c_{t-1} + \frac{1}{1+\varphi/\gamma} \bar{E}_t[c_{t+1}] - \frac{(1-\varphi/\gamma)}{1+\varphi/\gamma} \bar{E}_t[r_t - \pi_{t+1} - (x^c_{t+1} - x^c_t)]$$

This is the same Euler equation of Smets and Wouters (2003) with two key differences due to the presence of informational frictions.

First, expectations are heterogeneous, and hence average expectation replaces the full information expectation operator. Second, current endogenous variables and structural shocks are not observed. Thus, their expectations arise in optimal choices. In this case, households must form expectations about the current nominal rate, r_t, and the current preference shock, x^c_t.

Now consider the New Keynesian Phillips (NKPC) curve given by

$$\pi_t = \xi_p \iota_p \pi_{t-1} + \kappa_p \xi_p \bar{E}_t \left[mc_t + x^p_t\right] + \psi_p \bar{E}_t \left[\pi_t\right] + (1-\xi_p)\beta\xi_p \int_0^1 E_{it}[p^*_{i,t+1}]di \tag{35}$$

where $\psi_p \equiv (1-\xi_p)(1-\iota_p\beta\xi_p)$.

An additional difference appears in the NKPC as the average expectation of firms' future *own optimal prices* matters for inflation. Note that $\int_0^1 E_{it}[p^*_{i,t+1}]di \neq \bar{E}_t\left[p^*_{t+1}\right]$, since the law of iterated expectations does not hold for average expectations (Morris and Shin; 2005). Thus, we cannot write the last term in terms of the average expected future *aggregate optimal price*, which is related directly to future inflation.

As pointed out by Nimark (2008), as inflation depends on its own average expectation, higher-order expectations matter for inflation dynamics. By taking the average expectations of this equation and substituting it iteratively, we obtain that

$$\pi_t = \frac{\xi_p \iota_p}{1-\psi_p}\pi_{t-1} + \kappa_p\xi_p \sum_{k=1}^{\infty} \psi_p^{k-1} \bar{E}^{(k)}_t[mc_t + x^p_t] + (1-\xi_p)\beta\xi_p \sum_{k=0}^{\infty} \psi_p^k \bar{E}^{(k)}_t \left[\int_0^1 E_{it}[p^*_{i,t+1}]di\right], \tag{36}$$

where $\bar{E}^{(k)}_s[z_t] \equiv \int_0^1 E_{is}\left[\bar{E}^{(k-1)}_s[z_t]\right] di$, for any variable z_t, all $k \geq 1$ and $s \leq t$, with the convention that $\bar{E}^{(0)}_t[z_t] \equiv z_t$ and $\bar{E}^{(1)}_t[z_t] \equiv \bar{E}_t[z_t]$.

Firms must form not only beliefs about marginal cost, price mark-up shock, and future own price but also higher-order expectations about these variables. $\psi_p \in (0,1)$ is a key parameter that determines how strong the dependence of inflation on higher-order expectations is. One key feature is that inflation dependence on higher-order expectations decreases with the order.

The latter differs from Nimark's (2008) NKPC in two key aspects. First, our model includes partial indexation to past prices, which generates a backward term. Interestingly, ψ_p is decreasing in the indexation parameter, ι_p, i.e., indexation decreases the relevance of higher-order expectations. The intuition is the following. As non-optimizing firms index their prices to past inflation, optimal prices are less responsive to expectations about inflation. Thus, inflation is less dependent on higher-order expectations.

Second, it explicitly considers that $\int_0^1 E_{it}[p^*_{i,t+1}]di \neq \bar{E}_t\left[p^*_{t+1}\right]$, which prevents representing inflation as a function of the future inflation as in the full-information benchmark.[8]

One key feature of the solution method discussed in the next section is that we do not need to represent equilibrium conditions in terms of higher-order expectations as in equation (36) to solve the model. We can find a policy function that depends on higher-order expectations even defining equilibrium conditions in the system of equation in terms of average expectations as in equation (35).

3 Solution method

In this section, we present a novel solution method for DSGE models featuring imperfect and dispersed information.

The log-linearized equilibrium equations (65) can be written as the following general system of linear rational equations

$$\begin{gathered} F_1\bar{E}_t[Y_{t+1}] + F_2\int_0^1 E_{it}[Y_{i,t+1}]di + G_1Y_t + G_2\bar{E}_t[Y_t] + HY_{t-1} + \\ L\bar{E}_t[x_{t+1}] + M_1x_t + M_2\bar{E}_t[x_t] = 0_{m\times 1}, \end{gathered} \tag{37}$$

where $i \in [0,1]$ indexes an agent, $Y_{i,t}$ is a $m \times 1$ vector of individual choices, Y_t is a $m \times 1$ vector of aggregate endogenous variables and x_t is a $n \times 1$ vector of unobservable exogenous shocks.

Structural shocks x_t follow a stationary stochastic process

$$x_t = A_1x_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0, \Sigma_\varepsilon) \tag{38}$$

where A_1 is a diagonal matrix storing the persistence of each shock, and Σ_ε is a diagonal covariance matrix. Idiosyncratic signals are given by

$$s_{i,t} = C_xx_t + Dv_{i,t}, \quad v_{i,t} \sim \mathcal{N}(0, \Sigma_v), \tag{39}$$

[8] As pointed out in Appendix D of Angeletos and Huo (2021), Nimark (2008) and Melosi (2017) abstract in their derivation that $\int_0^1 E_{it}[p^*_{i,t+1}]di \neq \bar{E}_t\left[p^*_{t+1}\right]$. While this may not be quantitatively relevant, it is important for correctly defining the general system of equilibrium conditions (37) and the guessed policy function (40).

where Σ_v is a diagonal covariance matrix.[9]

Define the k-th order average expectation $E_t^{(k)}[\cdot]$ given information of period t about the vector of unobservable aggregate shocks x_t as $E_s^{(k)}[x_t] \equiv \int_0^1 E_{is}\left[E_s^{(k-1)}[x_t]\right] di$, for all $k \geq 1$ and s, t, with the convention that $E_t^{(0)}[x_t] \equiv x_t$ and $E_t^{(1)}[x_t] \equiv \bar{E}_t[x_t]$.

Following Nimark (2008), it is useful to denote the expectations hierarchy of x_t from order l to s as the vector

$$x_t^{(l:s)} \equiv \left[\begin{array}{cccc} E_t^{(l)}\left[x_t\right]' & E_t^{(l+1)}\left[x_t\right]' & \cdots & E_t^{(s)}\left[x_t\right]' \end{array} \right]',$$

for $s > l \geq 0$.

Following the insight of Nimark (2008), the solution method relies on a truncation of the state space to circumvent the problem of the infinite regression of expectations (see Townsend; 1983).[10] Specifically, we guess that the individual and aggregate equilibrium law of motion and expectation hierarchy are given by

$$Y_{i,t} = \mathbf{R}Y_{i,t-1} + \mathbf{Q_0}x_t + \mathbf{Q_1}E_{i,t}\left[x_t^{(0:\bar{k})}\right], \tag{40}$$

$$Y_t = \mathbf{R}Y_{t-1} + \mathbf{Q}x_t^{(0:\bar{k})} \tag{41}$$

$$x_t^{(0:\bar{k})} = \mathbf{A}x_{t-1}^{(0:\bar{k})} + \mathbf{B}\varepsilon_t, \tag{42}$$

where $(\mathbf{R}, \mathbf{Q_0}, \mathbf{Q_1}, \mathbf{A}, \mathbf{B})$ are finite dimensional matrices to be determined in equilibrium. $\mathbf{Q} = \mathbf{Q_0}e_x + \mathbf{Q_1}T$, e_x is the selection matrix such that $x_t = e_x x_t^{(0:\bar{k})}$ and T is a order transformation matrix such that $E_t^{(1)}\left[x_t^{(0:\bar{k})}\right] = Tx_t^{(0:\bar{k})}$.[11]

The system of equations (37) generalizes the existing methods by allowing: i) average expectation about future own variables (the term post multiplying F_2) and ii) endogenous state variables (the term post multiplying H). The latter allows solving medium-scale DSGE models such as the one in section 2.

Proposition 1 shows the dynamics for expectation hierarchy and the expressions for the fixed-point solution for $(\mathbf{A}, \mathbf{B})$.

Proposition 1. *Suppose that x_t is a stationary process, the expectation hierarchy follow equation (42) and agents use information from signals (39). Assuming common knowledge of rationality,*

[9]The model from Section 2 implies a diagonal structure for A_1, Σ_ε and Σ_v. The solution method does not require these assumptions.

[10]Nimark (2017) shows that as long as the impact on equilibrium outcomes of higher-order expectations decreases with the order, there exists a $\bar{k}$ such that the approximation error of the solution is less than any $\epsilon > 0$.

[11]See details in Appendix F

the individual and average expectations about the hierarchy are given by

$$E_{it}\left[x_t^{(0:\bar{k})}\right] = (I_k - \mathbf{K}C)\,\mathbf{A}E_{i,t-1}\left[x_{t-1}^{(0:\bar{k})}\right] + \mathbf{K}C\mathbf{A}x_{t-1}^{(0:\bar{k})} + \mathbf{K}C\mathbf{B}\varepsilon_t + \mathbf{K}Dv_{it}$$
$$\bar{E}_t\left[x_t^{(0:\bar{k})}\right] = \left(I_k - \bar{\mathbf{K}}C\right)\mathbf{A}\bar{E}_{t-1}\left[x_{t-1}^{(0:\bar{k})}\right] + \bar{\mathbf{K}}C\mathbf{A}x_{t-1}^{(0:\bar{k})} + \bar{\mathbf{K}}C\mathbf{B}\varepsilon_t.$$

where $C = C_x e_x$.

The expectation hierarchy consistent with the expectations above has coefficients $(\mathbf{A}, \mathbf{B})$ *that satisfy the matrix equations:*

$$\begin{gathered}\left(I_k - T'\bar{\mathbf{K}}C\right)\mathbf{A} = e_x' A_1 e_x + T'\left(I_k - \bar{\mathbf{K}}C\right)\mathbf{A}T \\ \left(I_k - T'\bar{\mathbf{K}}C\right)\mathbf{B} = e_x'\end{gathered} \tag{43}$$

such that $\bar{\mathbf{K}}$ *and* $\bar{\mathbf{P}}$ *solve the Riccati equation resulting from equations*

$$\begin{aligned}\bar{\mathbf{K}} &= \bar{\mathbf{P}}C'\left[C\bar{\mathbf{P}}C' + D\Sigma_v D'\right]^{-1} \\ \bar{\mathbf{P}} &= \mathbf{A}\left[\bar{\mathbf{P}} - \bar{\mathbf{K}}C\bar{\mathbf{P}}\right]\mathbf{A}' + \mathbf{B}\Sigma_\varepsilon\mathbf{B}'\end{aligned} \tag{44}$$

The Ricatti equation resulting from equations (44) solves the steady-state mean square error of the expectation hierarchy. The equations uses the standard Kalman Filter with hierarchy dynamics (42) as state equation and measurement equation as signals (39).

The key difference is that $(\mathbf{A}, \mathbf{B})$ is a result from agents signal extraction. Thus, the hierarchy persistence $(\mathbf{A})$ and response to shocks $(\mathbf{B})$ depend on the Kalman gain matrix $\bar{\mathbf{K}}$. As any signal extraction, the $\bar{\mathbf{K}}$ depends on the unobserved state dynamics $(\mathbf{A}, \mathbf{B})$.

Therefore, the equilibrium solution $(\mathbf{A}, \mathbf{B}, \mathbf{K}, \mathbf{P})$ is the fixed point from equations (43-44).

Since the signals (39) are exogenous, they do not depend on equilibrium conditions (37). This simplifies finding the guessed coefficients for the equilibrium $(\mathbf{R}, \mathbf{Q_0}, \mathbf{Q_1})$ and tremendously diminishes the computational costs.[12]

Before presenting the equilibrium dynamics in Proposition below, we want to emphasize one assumption that simplifies substantially this task. We assume that when forming expectations about Y_t using the guessed solutions (40-41), agents know the past value of endogenous variables, Y_{t-1}.

This assumption contrasts with the expectation formation from Proposition 1 in which expectations depend only on exogenous signals. This avoid learning from (past) endogenous variables,

[12]For instance, Melosi (2017) explains in his replication files that: "Be aware that estimating the DIM may take several weeks or even a few months depending on the computer used for this task." In our application, which has a considerable higher scale model, takes 36 hours for 400k posterior draws in a standard notebook.

which generates a feedback between the signal extraction and endogenous variable responses to the hierarchy.[13] Ribeiro (2017) considers the general case with endogenous signals.

This assumption is implicit in standard solution methods for imperfect under common information such as Blanchard et al. (2013) and Baxter et al. (2011). This is also a common assumption when studying the New Keynesian Phillips Curve under dispersed information (e.g., Angeletos and La'O; 2009; Angeletos and Huo; 2021).

Proposition 2. *For a given dynamics expectation hierarchy from equation (42), the system of equations (37) has a recursive equilibrium law of motion (41) whose matrix coefficients satisfy:*

1. $(\mathbf{R}, \mathbf{Q_0}, \mathbf{Q_1})$ *satisfy the matrix equations:*

$$F\mathbf{R}^2 + G\mathbf{R} + H = 0_{m\times m} \tag{45}$$

$$\left[F_1\mathbf{R} + G_1\right]\mathbf{Q_0} + M_1 = 0_{m\times n} \tag{46}$$

$$\begin{aligned}&\left[F_1\mathbf{R} + G_1\right]\mathbf{Q_1} + F_1\mathbf{Q_1}\mathbf{A} + (F_2\mathbf{R} + G_2)\mathbf{Q_1}T + F_2\mathbf{Q_1}T\mathbf{A} + \\ &\left[(F_2\mathbf{R} + G_2)\mathbf{Q_0} + F\mathbf{Q_0}A_1 + (LA_1 + M_2)\right]e_x = 0_{m\times k}\end{aligned} \tag{47}$$

   where $F \equiv F_1 + F_2$, $G \equiv G_1 + G_2$ *and* $k = n(\bar{k}+1)$.

2. *(Uhlig; 2001)* $\mathbf{R}$ *has a unique stable solution if all eigenvalues of* $\mathbf{R}$ *are smaller than unity in absolute value.*

3. *Given the solution of* $\mathbf{R}$*, denote the matrices* (V_0, V_1) *such that:*

$$V_0 = F_1\mathbf{R} + G_1 \tag{48}$$

$$V_1 = I_k \otimes (F_1\mathbf{R} + G_1) + T' \otimes (F_2\mathbf{R} + G_2) + \mathbf{A}' \otimes F_1 + T'\mathbf{A}' \otimes F_2 \tag{49}$$

   Provided that there exists a inverse for V_0 *and* V_1*, the equilibrium solution for* $(\mathbf{Q_0}, \mathbf{Q_1})$ *is given by*

$$\mathbf{Q_0} = -V_0^{-1}M_1 \tag{50}$$

$$vec(\mathbf{Q_1}) = -V_1^{-1}vec\left(\left[(F_2\mathbf{R} + G_2)\mathbf{Q_0} + F\mathbf{Q_0}A_1 + (LA_1 + M_2)\right]e_x\right) \tag{51}$$

   where $vec(\cdot)$ *denotes columnwise vectorization.*

Given the individual response to shocks $(\mathbf{Q_0})$ *and to the individual expectation about the hierarchy* $(\mathbf{Q_1})$*, the aggregate response to the hierarchy is given by:* $\mathbf{Q} = \mathbf{Q_0}e_x + \mathbf{Q_1}T$

[13] One can think this assumption as a deviation from rationality as agents do not incorporate that their knowledge about Y_{t-1} can be used to learn about $x_{t-1}^{(0,\bar{k})}$, which would also change their beliefs about $x_t^{(0,\bar{k})}$.

Proposition 2 connects the solution of imperfect common knowledge models with the standard undetermined coefficients solution for full information DSGE models.

Condition (45) is exactly the same as the usual "brute force" approach of Uhlig (2001). In other words, he equilibrium $\mathbf{R}$ is the same that would happen if agents had full information, as shown in Appendix F.[14]

This has two key implications. First, $\mathbf{R}$ can be computed with standard techniques. Second, informational frictions does not affect how endogenous variables respond to state variables. This does not imply that information frictions are not important their persistence. Endogenous variables persistence also reflects the hierarchy persistence.

Our solution method is a natural extension of Uhlig's (2001) full information and Blanchard et al.'s (2013) imperfect information methods. It is also related to the dispersed information method from Melosi (2017). The key difference is that his method abstracts from state endogenous variables but allows for endogenous signals.

Appendix F further explore the relation with those methods and show the details of the algorithm to find the fixed point solution from Proposition 1.

4 Estimation

In this section, we discuss the data, the prior choices, and the posterior results of the Bayesian estimation. We estimate the model featuring dispersed information ('DI model') using two different datasets: one containing macroeconomic data only and other including both macroeconomic and expectation data. For comparison, we also estimate the same model under full-information ('FI model') where we abstract from information frictions in the same datasets.

4.1 Data

We collect U.S. quarterly macroeconomic and expectation data from 4Q:1981 to 4Q:2007. The data on macroeconomic aggregates include the series of real GDP, real consumption, real investment, GDP deflator index, annualized Fed funds rate, hours worked index and nominal wages index, and come from the Federal Reserve Database (FRED).

We also gather the average expectation about one-quarter ahead real GDP, real consumption and real investment, inflation and three-month Treasure bill rate from the US Survey of Professional Forecasters (SPF). A detailed description of the datasets can be found in Appendix B.

When estimating medium scale DSGE models, the use of macroeconomic data is quite standard in the literature. However, the inclusion of expectation data is notably less common. In recent

[14]The key assumption for this result is that agents observe past endogenous variables at period t.

years, several papers have tried to incorporate expectation data to discipline macroeconomic models with complete and incomplete information. Del Negro and Eusepi (2011) adds one-year ahead inflation expectations to estimate a conventional New Keynesian model. Melosi (2017) estimates a small scale dispersed information model using data on inflation forecasts one-quarter and one-year ahead. Del Negro et al. (2015) uses ten-year inflation expectations from the Blue Chip and SPF.

In this paper, we divert from previous literature in two ways. First, we use expectation data on a broader set of variables. Second, we use data on *current forecast revisions* (the nowcast minus last period forecast). The reasoning for the latter is as follows.

By aggregating equation (32), one can see that

$$\bar{Rev}_{t|t-1}[x^l_t] \equiv \bar{E}_t[x^l_t] - \bar{E}_{t-1}[x^l_t] = \bar{k}_l \left[s^l_t - \bar{E}_{t-1}[s^l_t] \right] = \bar{k}_l \bar{Fe}_{t-1}[x^l_t] \tag{52}$$

where $s^l_t = \int_0^1 s^l_{it} di = x^l_t$ is the aggregate signal, which due to the simply signal structure (30) is equal to the shock l. $\bar{Rev}_{t|s}[\cdot]$ denotes the average expectation revision from period t in comparison with forecast from period s and $\bar{Fe}_s[x^l_t] = x^l_t - \bar{E}_s[x^l_t]$ denotes the forecast error of a forecast made in period s.

Note that the current forecast revision of shock l depends on the information frictions and signals only.

By iterating equation (34), one can see that forecast $\bar{E}_t[x^l_t]$ depends on the whole history of previous signals, the information frictions and shock's persistence. Longer forecast horizons ($\bar{E}_t[x^l_{t+h}]$) and revisions ($\bar{Rev}_{t|t-1}[x^l_{t+h}]$) are likely to include more confounding variability as it depends more heavily on the shocks persistence.[15]

The discussion above is true if we had data on the expectations about shocks. In practice, we have data on endogenous variables, which creates another layer of complication.

Considering this issues, in our view, short horizon forecast revisions provide the best representation of expectation data for the purpose of identifying the degree of informational friction.

For completeness, we also estimate the model using two different expectational variables: i) one-quarter ahead forecast errors; and ii) one-quarter ahead forecasts. We compare those estimates with the baseline and evaluate which dataset provide better information about informational frictions.

[15] The h-step ahead forecast is given by $\bar{E}_t[x^l_{t+h}] = \rho^h_l \bar{E}_{t|}[x^l_t]$ and analogous revision is $\bar{Rev}_{t|t-1}[x^l_{t+h}] = \rho^h_l \bar{Rev}_{t|t-1}[x^l_t]$. They also depend on shocks' persistence, ρ_l and the horizon h.

4.2 Measurement equations

The measurement equations that link data with the variables for both models are the following

$$
\begin{aligned}
dy_t^{obs} &= \bar{\gamma} + y_t - y_{t-1} \\
dc_t^{obs} &= \bar{\gamma} + c_t - c_{t-1} \\
di_t^{obs} &= \bar{\gamma} + i_t - i_{t-1} \\
dw_t^{obs} &= \bar{\gamma} + w_t - w_{t-1} \\
l_t^{obs} &= l_t \\
\pi_t^{obs} &= \bar{\pi} + 4\pi_t \\
r_t^{obs} &= \bar{r} + 4r_t \\
rev_{dy,t|t-1}^{obs} &= \bar{E}_t[\Delta y_t] - \bar{E}_{t-1}[\Delta y_t] + \varepsilon_{dy,t}^{me} \\
rev_{dc,t|t-1}^{obs} &= \bar{E}_t[\Delta c_t] - \bar{E}_{t-1}[\Delta c_t] + \varepsilon_{dc,t}^{me} \\
rev_{di,t|t-1}^{obs} &= \bar{E}_t[\Delta i_t] - \bar{E}_{t-1}[\Delta i_t] + \varepsilon_{di,t}^{me} \\
rev_{\pi,t|t-1}^{obs} &= 4\left(\bar{E}_t[\pi_t] - \bar{E}_{t-1}[\pi_t]\right) + \varepsilon_{\pi,t}^{me} \\
rev_{r,t|t-1}^{obs} &= 4\left(\bar{E}_t[r_t] - \bar{E}_{t-1}[r_t]\right) + \varepsilon_{r,t}^{me},
\end{aligned}
\tag{53}
$$

where $\bar{\gamma} \equiv 100(\gamma - 1)$ is the quarterly trend growth rate of productivity. The variables with superscript "*obs*" correspond to the variables in the database. The notation dx_t^{obs} denotes the quarterly growth rate of the variables $x \in \{y, c, i, w\}$ at period t. The data on hours worked (l_t^{obs}) is demeaned, while inflation (π_t^{obs}) and nominal interest rate (r_t^{obs}) are computed in annual terms. $\bar{\pi} \equiv 400(\bar{\Pi} - 1)$ and $\bar{r} \equiv 400(\bar{R} - 1)$ stand for the steady-state annualized inflation and nominal interest rates, respectively. The variables $rev_{\pi,t|t-1}^{obs}$, for $x \in \{dy, dc, di, \pi, r\}$, are the current forecast revision.

Measurement errors ε_x^{me} in the observational equations associated with forecast errors are introduced to avoid stochastic singularity.[16] They also capture the fact that the expectation data from SPF are not completely consistent with the average expectations of the agents populating our model. Measurement errors follow an i.i.d. Gaussian distribution with standard deviations denoted by σ_x^{me}, with $x \in \{dy, dc, di, \pi, r\}$.

The measurement equations when using data on forecasts or forecast errors are shown in Appendix C. Those specification change revision data for either forecast or forecast errors with their own measurement errors and links model variables.

[16] This is required since the complete dataset includes 12 time series, while the model has only 7 shocks.

4.3 Prior distributions

As standard in the literature, we fix the value of some parameters that are not easily identified. The quarterly depreciation rate δ is set to 2.5%, the value widely used in the literature.[17] The ratio of government expenditures to output g_y is calibrated to its historical mean value of 18%. The steady-state wage markup $\bar{\mu}^w$ is fixed at 1.5 as in Smets and Wouters (2007). Finally, the capital share of the production function α is fixed 0.3 to be consistent with the historical capital-output ratio.

The introduction of informational frictions is likely to affect the estimation of real and nominal frictions, so we do not rely completely on estimates from previous papers to choose priors. Instead, most of the priors are relatively loose. Table 1 shows the choices for priors on the left side. The priors are the same for the FI and DI models. Obviously, the full information model does not include the informational friction parameters and estimates with macroeconomic data only do not include measurement errors for revision data.

The prior for parameters that have support between 0 and 1, such as the persistence of shocks, partial adjustment of the Taylor rule, the Calvo frictions on prices and wages, indexation of prices and wages to past inflation and habit persistence in consumption, follow a beta distribution ($\mathcal{B}$) with a mean of 0.5 and standard deviation of 0.2.

The trend growth, inflation rate, and nominal interest rate steady-state parameters have priors with gamma distribution ($\mathcal{G}$) whose mean equals to their sample average. The standard deviations are 0.50 for the first prior and 1.00 for the others in order to ensure fairly loose priors.

The priors on the adjustment cost of investment, capital utilization, and the inverse of Frisch elasticity are taken from Del Negro et al. (2007). The parameters specifying the Taylor rule have priors with gamma distribution whose mean values are $\phi_\pi = 1.5$ and $\phi_y = 0.2$ and the standard deviations are 0.5 and 0.1, respectively.

Priors for the standard deviation of the structural shocks are distributed as an inverse gamma ($\mathcal{IG}$) with means chosen to match standard deviations of observables in the pre-sample 1957Q1-1981Q1 as in Del Negro and Eusepi (2011). Standard deviations are set to 1.00, which implies fairly loose priors.

Priors for Kalman gains of each shock are set to follow a beta distribution with a mean of 0.5 and a standard deviation of 0.2. The value of 0.5 is similar to estimates from Coibion and Gorodnichenko (2015) using one-year inflation expectations and Del Negro and Eusepi (2011) for one-year data on output, consumption, investment, inflation, and nominal rates. Note that their estimates of frictions are using macroeconomic aggregates and not actual shocks. We discuss more on this in section 5.

There is almost no evidence of the degree of informational friction for each shock. One exception

[17] See for instance Christiano et al. (2005), Smets and Wouters (2007) and Del Negro et al. (2007).

is Melosi (2017) which finds that informational frictions on expenditure shocks are substantial in comparison with productivity shocks. Since his model has a small scale and few shocks, we pursue an agnostic view about which shocks are more prone to informational frictions.

We use the Kalman gain ($\bar{k}_l$) directly as a parameter instead of the noise-to-signal ratio ($r_l = \tau_l/\sigma_l$) as Del Negro and Eusepi (2011) or the private signal variances (τ_l) as Melosi (2017). As discussed by Del Negro and Eusepi (2011), forming priors on r_l of a shock implicitly forms a prior for the private signal's variance dependent on the shock's variance, σ_l. Analogously, when placing priors on the Kalman gains of each shock, it implicitly generates a prior for the NRS dependent on the shock's persistence, ρ_l (see equation 33).[18]

Therefore, despite the shock processes (ρ_l, σ_l) having different priors for each shock l, when assuming the same prior for the Kalman gains of all shocks, there is an implicit prior for $\tau_l|\rho_l, \sigma_l$ that allows the same information friction for all shocks a priori.

Finally, priors for the standard deviation of measurement errors on the time series of forecast errors are distributed as inverse gamma with a mean that matches roughly 5% of the total standard deviation of each revision data. The standard deviations are small to ensure very tight priors, which avoids measurement errors to explain more than 20% of the variation in the revision data.

4.4 Posterior distributions

Bayesian techniques are applied to estimate the models, by combining the prior densities described before with the likelihood function computed using the Kalman filter. We use the Random Block Random-Walk Metropolis-Hastings (RB-RWMH) algorithm with five blocks to draw from the posterior distribution.[19] Table 1 reports the posterior estimates for both models (FI and DI models) for the Revision dataset. For each model and dataset, the table shows the mean and the 5%-95% percentiles of the posterior density.

Several interesting results arise from our estimation exercise. When we include expectation data in the estimation, we find that parameter estimates for both models change substantially. Under the complete dataset, the estimated persistence of several shocks declines substantially, in particular for the monetary, investment and preference shocks. Moreover, the adjustment cost in investment rises significantly (more than three times than standard estimates from SW), and the Taylor-rule response to inflation and past interest rates both increase. Finally, in the DI model, including data on forecast revisions increases both the estimated levels of information frictions and the variances of the shocks.

[18] Del Negro and Eusepi (2011) has also a robustness check that they estimate the Kalman gain directly without taking into account its relationship with other parameters.

[19] Estimates from RB-RWMH have substantially better chain convergence diagnostics than standard RWMH as they generate chains with lower autocorrelation. We use 400k draws and a 50% burn-in. The candidate distribution is a multivariate normal with

Table 1: Prior and Posterior distributions

				Dataset: Macroeconomic data only						Dataset: Macroeconomic and expectations data					
	Prior			Posterior: FI model			Posterior: DI model			Posterior: FI model			Posterior: DI model		
	Dist.	Mean	Std. Dev.	Mean	5%	95%	Mean	5%	95%	Mean	5%	95%	Mean	5%	95%
Endogenous propagation parameters															
χ	$\mathcal{G}$	2	0.75	2.24	1.40	3.40	1.85	1.17	2.92	2.24	1.59	3.04	1.84	0.99	3.04
φ	$\mathcal{B}$	0.5	0.2	0.96	0.90	0.99	0.33	0.13	0.53	0.98	0.96	0.99	0.98	0.96	0.99
a''	$\mathcal{G}$	0.2	0.1	0.39	0.23	0.59	0.31	0.17	0.49	0.46	0.29	0.69	0.46	0.29	0.67
s''	$\mathcal{G}$	4	1.5	3.59	1.63	6.25	5.14	2.52	8.39	16.32	12.39	20.87	12.35	8.78	16.45
ϕ_y	$\mathcal{G}$	0.2	0.1	0.39	0.25	0.56	0.20	0.12	0.29	0.06	0.03	0.10	0.31	0.21	0.42
ϕ_π	$\mathcal{G}$	1.5	0.5	0.77	0.45	1.18	1.01	0.62	1.52	2.77	2.29	3.37	1.55	0.99	2.18
ϕ_r	$\mathcal{B}$	0.5	0.2	0.71	0.60	0.80	0.71	0.60	0.82	0.88	0.85	0.91	0.92	0.89	0.95
ι_p	$\mathcal{B}$	0.5	0.2	0.22	0.04	0.76	0.22	0.04	0.67	0.04	0.01	0.08	0.69	0.48	0.80
ι_w	$\mathcal{B}$	0.5	0.2	0.35	0.10	0.67	0.48	0.15	0.82	0.46	0.16	0.78	0.39	0.12	0.72
ξ_p	$\mathcal{B}$	0.5	0.2	0.98	0.96	0.99	0.98	0.97	1.00	0.88	0.85	0.91	0.98	0.98	0.99
ξ_w	$\mathcal{B}$	0.5	0.2	0.81	0.70	0.89	0.08	0.02	0.18	0.80	0.74	0.85	0.92	0.85	0.96
Steady-state parameters															
π_{ss}	$\mathcal{G}$	2.62	0.5	2.20	1.68	2.67	2.29	1.80	2.70	2.81	2.37	3.25	1.49	1.06	1.96
r_{ss}	$\mathcal{G}$	5.15	0.5	6.96	6.10	7.87	7.38	6.38	8.41	6.22	5.52	6.95	7.44	6.38	8.57
γ_{ss}	$\mathcal{G}$	0.5	0.1	0.36	0.31	0.41	0.40	0.35	0.45	0.35	0.32	0.37	0.47	0.45	0.50
Structural shocks parameters															
ρ_c	$\mathcal{B}$	0.5	0.2	0.42	0.22	0.68	0.89	0.79	0.96	0.24	0.15	0.34	0.13	0.07	0.20
ρ_i	$\mathcal{B}$	0.5	0.2	0.52	0.36	0.70	0.48	0.36	0.60	0.25	0.19	0.32	0.29	0.22	0.35
ρ_g	$\mathcal{B}$	0.5	0.2	0.98	0.97	1.00	0.99	0.97	1.00	0.98	0.96	0.99	0.99	0.98	1.00
ρ_p	$\mathcal{B}$	0.5	0.2	0.67	0.15	0.84	0.55	0.13	0.77	0.88	0.83	0.91	0.10	0.03	0.21
ρ_w	$\mathcal{B}$	0.5	0.2	0.33	0.16	0.53	0.88	0.79	0.96	0.28	0.15	0.43	0.34	0.21	0.48
ρ_r	$\mathcal{B}$	0.5	0.2	0.96	0.91	0.98	0.91	0.79	0.98	0.32	0.23	0.41	0.41	0.29	0.54
ρ_a	$\mathcal{B}$	0.5	0.2	0.92	0.87	0.97	0.96	0.92	0.99	0.88	0.84	0.92	0.998	0.995	0.999
σ_c	$\mathcal{IG}$	0.3	1	0.16	0.11	0.22	0.14	0.08	0.26	0.20	0.17	0.23	1.72	1.28	2.29
σ_i	$\mathcal{IG}$	1.5	1	1.25	1.02	1.50	1.72	1.21	2.38	1.26	1.10	1.44	5.65	4.63	6.84
σ_g	$\mathcal{IG}$	0.3	1	0.53	0.46	0.61	0.53	0.46	0.62	0.60	0.53	0.68	0.50	0.43	0.58
σ_p	$\mathcal{IG}$	0.1	1	0.06	0.03	0.15	0.10	0.05	0.20	0.04	0.03	0.05	0.39	0.31	0.50
σ_w	$\mathcal{IG}$	0.75	1	0.54	0.44	0.65	0.64	0.47	0.91	0.57	0.47	0.68	0.91	0.55	1.50
σ_r	$\mathcal{IG}$	0.1	1	0.13	0.11	0.15	0.12	0.11	0.13	0.13	0.11	0.14	0.12	0.11	0.14
σ_a	$\mathcal{IG}$	0.3	1	0.41	0.37	0.46	0.42	0.37	0.47	0.41	0.37	0.46	0.44	0.39	0.49
Information friction parameters[a]															
$\bar{k}_c$	$\mathcal{B}$	0.5	0.2				0.67	0.50	0.82				0.25	0.20	0.31
$\bar{k}_i$	$\mathcal{B}$	0.5	0.2				0.84	0.70	0.95				0.40	0.36	0.45
$\bar{k}_g$	$\mathcal{B}$	0.5	0.2				0.60	0.47	0.73				0.52	0.49	0.55
$\bar{k}_p$	$\mathcal{B}$	0.5	0.2				0.83	0.64	0.96				0.39	0.30	0.47
$\bar{k}_w$	$\mathcal{B}$	0.5	0.2				0.77	0.57	0.93				0.68	0.41	0.90
$\bar{k}_r$	$\mathcal{B}$	0.5	0.2				0.20	0.07	0.36				0.72	0.63	0.80
$\bar{k}_a$	$\mathcal{B}$	0.5	0.2				0.90	0.79	0.97				0.05	0.04	0.07
Measurement errors standard deviations[b]															
$\sigma^{rev}_{me,y}$	$\mathcal{IG}$	0.1	0.02							0.47	0.41	0.53	0.29	0.26	0.33
$\sigma^{rev}_{me,c}$	$\mathcal{IG}$	0.1	0.02							0.42	0.38	0.48	0.21	0.18	0.23
$\sigma^{rev}_{me,i}$	$\mathcal{IG}$	0.5	0.2							2.24	2.00	2.52	1.22	1.09	1.37
$\sigma^{rev}_{me,\pi}$	$\mathcal{IG}$	0.15	0.05							0.75	0.67	0.83	0.45	0.40	0.50
$\sigma^{rev}_{me,r}$	$\mathcal{IG}$	0.15	0.05							0.30	0.26	0.34	0.25	0.22	0.29
Marginal Likelihood				-915.50			-926.25			-1554.40			-1293.30		

Note: $\mathcal{G}$, $\mathcal{IG}$ and $\mathcal{B}$ stand for: Gamma, Inverse Gamma and Beta distributions, respectively.
[a]Applies to the dispersed information model only.
[b]Applies to the dataset including macroeconomic and expectations data only.

There are notable differences in the parameter estimates for the FI and DI models under both datasets. The introduction of information frictions substantially reduces habit formation and wage stickiness, which is particularly evident when models are estimated using macroeconomic data only. Second, there is a considerable decrease in the variance of measurement errors, a first indication that the DI model provides a better explanation for expectation data, which we confirm further when comparing the marginal likelihoods of the FI and DI models. Introducing information frictions also increases the variances of shocks for both datasets, with a more pronounced effect observed in the complete dataset. The higher shock volatility is somewhat offset by the impact of information frictions, resulting in less pronounced responses. Overall, the estimates derived from the complete dataset exhibit stronger standard frictions, including habits, adjustment costs, price and wage indexation, and robust informational frictions across all shocks. Monetary shocks display comparatively smaller information frictions, although they remain distant from the full information benchmark.

Overall, the estimates for the complete dataset has stronger standard frictions (habits, adjustment costs, price and wage indexation and rigidity) and also strong informational frictions for all shocks. Monetary shocks are the ones with smaller information frictions, but still far from the full information benchmark.

Table 1 shows the logarithm of the marginal likelihood for each model and dataset. When comparing the marginal likelihood of both models for the dataset containing only macroeconomic data, we find a difference of 11.5 log points in favor of the DI model. For the dataset including expectation data, the difference in favor of the DI model increases considerably to 261.1 log points. This result provides strong evidences that a model with informational frictions instead of the standard reduced-form frictions used to generate inertia in DSGE models can explain significantly better the expectation data.

5 Model evaluation and implications for measures of information frictions

Coibion and Gorodnichenko (2015) explore a general feature of models with information frictions that relate forecast errors and revisions to provide simple measures of informational frictions from the SPF.

In this section, we follow a similar empirical strategy. For each variable z in the SPF, we estimate the following regression

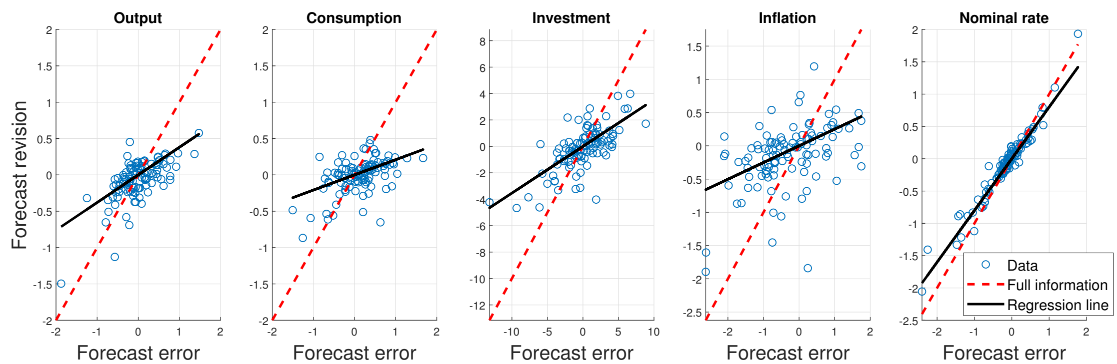
$$Rev_{t|t-1}[z_t] = \beta FE_{t-1}[z_t] + error_t, \tag{54}$$

where $\bar{Rev}_{t|t-1}[z_t] = \bar{E}_t[z_t] - \bar{E}_{t-1}[z_t]$ and $\bar{FE}_{t-1}[z_t] = z_t - \bar{E}_{t-1}[z_t]$

Following the same steps as (52) one can see that if z follows an AR(1), then β is equal to the Kalman gain, $\bar{k}_z$.

Figure 1 shows the scatter plot for each variable with the fitted line in solid black line. In the following, we refer to those estimates as empirical Kalman gains (eKGs), i.e., Kalman gains directly estimated in the data.

Figure 1: Empirical Kalman gains from SPF expectation data (1981Q4-2007Q4)



Note: Output, consumption and investment refers to their growth rates. Inflation and interest rates are in levels.

If agents had full information, all data points should be in the red dashed line (the 45º line). Thus, Figure 1 shows suggests a clear departure of FI in the SPF data similarly to previous studies such as CG and BGMS.

It is also clear that variables differ in their information friction measure. For instance, the Fed funds rate has forecast revisions much more aligned with the forecast errors than any other series.

In our model, the AR(1) assumption holds for the shocks' dynamics. Thus, regression (54) strictly applies only if we had data on shocks instead of endogenous variables. The equilibrium dynamics for endogenous variables (41) depends on their lags and the whole hierarchy of expectations about all shocks. Therefore, the relationship between forecast errors and forecast revisions from endogenous variables does not generate a close form prediction as in equation (54).

In a dynamic beauty context model with one action and $AR(1)$ fundamental, Angeletos and Huo (2021) show analytically that eKGs are determined by the interaction of informational frictions with other frictions in the model.

Therefore, in our general model, estimates from those regressions capture a combination of: i) informational frictions of each shock; ii) "ad-hoc" frictions associated with endogenous persistence.[20]

[20] CG extends their empirical specification to more general data generating processes such as $AR(p)$ for inflation or a $VAR(1)$ some selected macroeconomic models. In our model, the persistence of endogenous variables, R, from the solution (41) is a function of structural parameters.

The eKG of each variable is likely to be associated with different frictions. may capture information frictions but also many other frictions. With our estimated model, we can shed light on the relative importance of those frictions.

In the following, we do three experiments. First, we abstract from the issues implied by the model in those regressions and evaluate if data generated by the DI model generates estimates for Kalman gains that we find in the SPF data. This may be seem as an unmatched moment that is not used in the estimation that we want to check if the model is able to capture.

Second, we compare the eKGs from simulated data based on posterior means from *different expectation datasets*. In our baseline estimates, we use data on forecast revisions. We compare the implied eKGs from the baseline with alternative posterior estimates based on forecast errors and forecasts.

Third, we evaluate how sensitive eKGs are to changes in real, nominal and informational frictions. The latter pursue to evaluate if the empirical strategy proposed by CG truly uncovers informational frictions in our model.

5.1 Does the model match empirical measures of informational frictions?

In this section, we pursue the following experiment. Consider the vector of parameters, Θ, that collects all parameters of the model and the vector of some selected endogenous variables, z_t. For a given Θ, we simulate a n_{sim} samples of those variables and their expectations $\{z_t, \bar{E}_t[z_t], \bar{E}_{t-1}[z_t]\}$ and measurement errors $\{\varepsilon^{me}_{z,t}\}$ for $t = 1, \cdots, T_{sim}$ and $z \in \{dy, dc, di, \pi, r\}$. Then, we construct n_{sim} time-series for average forecast revisions and forecast errors such that

$$\bar{Rev}^{sim}_{z,t|t-1} = \bar{E}_t[z_t] - \bar{E}_{t-1}[z_t] + \varepsilon^{me}_{z,t}$$
$$\bar{Fe}^{sim}_{z,t-1} = z_t - \bar{E}_{t-1}[z_t]$$

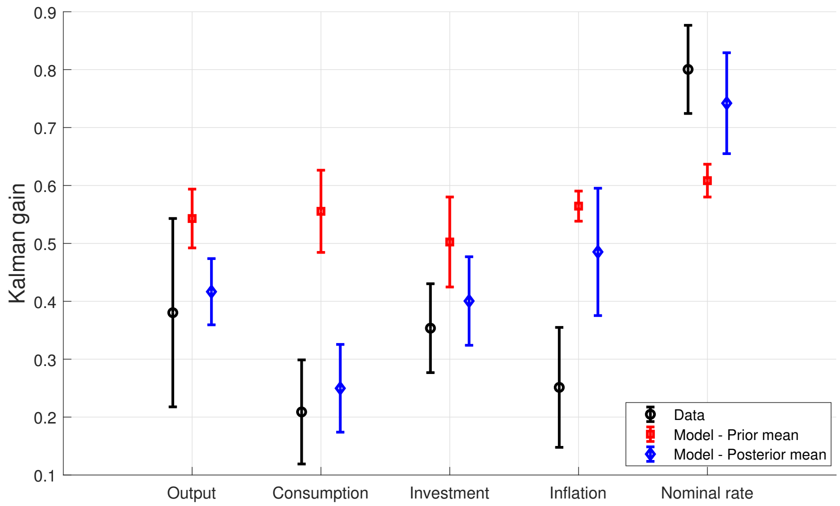
for $t = 1, \cdots, T_{sim}$ and $z \in \{dy, dc, di, \pi, r\}$. Note that the constructed series for revision is consistent with the measurement equations (53) as we also draw the measurement errors.

Then, we compute the implied eKGs for n_{sim} regressions (54) and get the average Kalman gain and standard deviations.

Figure 2 shows the results for $T_{sim} = 105$, $n_{sim} = 100$ and parameter vector equals to the prior mean ($\Theta = \underline{\Theta}$) and posterior means ($\Theta = \bar{\Theta}$).[21] The black error bars (with $\circ$) represent the estimates from the SPF presented in Figure 1.

[21] We use $T_{sim} = 105$ to match the sample size from SPF data. $n_{sim} = 100$ is sufficient large to avoid sampling variability affecting meaningfully the Kalman gain estimates.

Figure 2: Empirical Kalman gains from simulated data from DI model



Note: Output, consumption and investment refers to their growth rates. Inflation and interest rates are in levels.

The eKGs from simulated data using the priors mean (red error bar with $\square$) are similar for all variables and close to the structural Kalman gains (sKG, henceforth), $\bar{k}_x, x \in \{c, i, g, p, w, r, a\}$, whose mean is 0.5.

When comparing the same estimates from the posterior mean (blue error bar with $\diamond$), those are much more aligned with the direct estimates from the data. Except for inflation, the Kalman gain estimates from the model are very close to their data counterpart.

Recall that we use data from forecast revisions at the same horizon in the estimation, but the Bayesian estimates do not pursue to match this relation between forecast revisions and forecast errors.

5.2 Which dataset: Forecasts, forecast errors or forecast revisions?

This section compares estimates using different expectation datasets. In our baseline estimation, we use data on forecast revisions because our model suggests that this is a closer measure of information frictions and agents' information sets.

We reestimate the DI model by changing forecast revision data for two other alternatives: i) forecasts and ii) forecast errors (FE, henceforth). Appendix C provides details on measurement equations for each dataset. Each specification has its own measurement error for each expectational variable.

Interestingly, using data on forecasts or forecast errors is not equivalent. The likelihood depends on the *econometrician's* one-step ahead forecast errors. Thus, when using forecasts as data, forecast errors from *econometrician's information set* enter in the likelihood. This is different from the average forecast error from SPF's participants.

Table 3 in Appendix D shows the estimates from both datasets and compares them with two baselines: i) macroeconomic data only and ii) forecast revision dataset. Priors from parameters in this table are common and the same as in Table 1. Table 4 in the same Appendix shows the specific priors for measurement errors and their posteriors. Prior choices are made analogously to the baseline measurement error described in section 4.3.

When comparing the posterior estimates from different datasets, some results stand out. Estimates from information frictions differ remarkably depending on which expectation data is used. The FE dataset tend to less informational frictions when comparing with the baseline 'Revision dataset' (except for $\bar{k}_w$).

The ranking of which shock has stronger information frictions is also different. For instance, baseline estimates imply quite strong information frictions for TFP and preference shocks (0.05 and 0.25 respectively), while the opposite is true for the FE dataset (0.59 and 0.80 respectively). Estimates are similar only for $\bar{k}_r$ (roughly 0.7). The forecast dataset also does not provide similar results for sKGs. It differs markedly regarding the information friction for the monetary shock ($\bar{k}_r = 0.14$) and price markup ($\bar{k}_r = 0.87$).

Real frictions such as habits, utilization and investment adjustment costs are similar across estimates. They differ in terms of nominal rigidities. The baseline dataset implies quite high price and wage rigities and moderate indexation for both. FE dataset implies slightly lower estimates for price and wage rigidities whereas the Forecast dataset estimates imply moderate rigidity for prices and very weak for wages. Both datasets find that price indexation plays a minor role.

Since there is no clear pattern in informational frictions in those datasets, one question arises: which dataset provides more accurate information about those frictions?

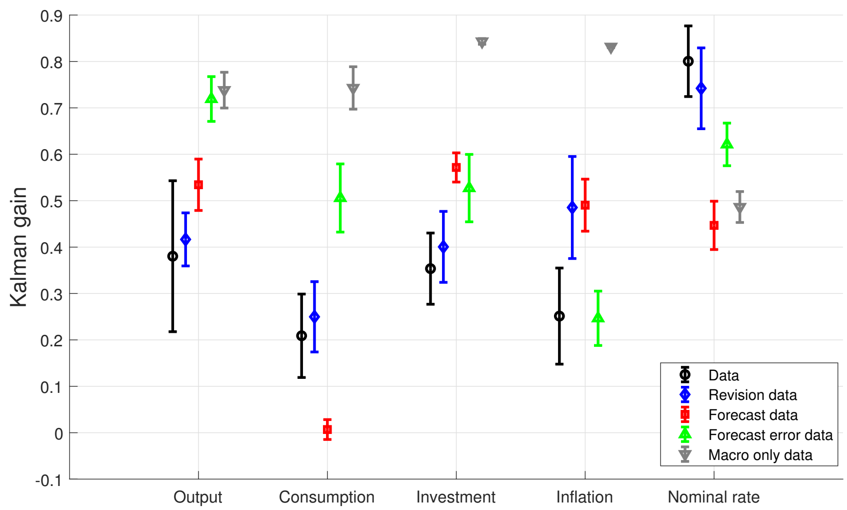
Since we are using different datasets, we cannot use marginal likelihoods to compare the fit of the alternative estimates. Instead, we use the eKGs from the SPF data to evaluate the alternative estimates. Specifically, we do the analogous experiment from the previous section, but using the posterior means from different datasets.

Figure 3 shows the eKGs from the alternative datasets and compares them to the estimates directly from the SPF. It also compares with estimates from simulated data using the posterior mean using macroeconomic data only.

When abstracting from using data on expectations (grey error bar with $\triangledown$), the estimates yield substantially less information frictions for output, consumption, investment and inflation; and higher information frictions for interest rate than estimated directly from SPF data (black error bar with $\circ$).[22] Recall that this dataset yields estimates for sKGs ranging from $0.6 - 0.9$ for all

[22]Estimates for 'Macroeconomic only dataset' have much lower error bands because the simulation does not

Figure 3: Empirical Kalman gains from simulated data from DI model



Note: Output, consumption and investment refers to their growth rates. Inflation and interest rates are in levels.

shocks except the monetary shock. The Kalman gain for the monetary shock, $\bar{k}_r$, is 0.20. When including revision data in the estimation, the contrary holds: estimate for $\bar{k}_r$ is 0.72 and for other shocks ranges from $0.05 - 0.68$. These sharp differences in sKGs explain the implied differences in eKGs.

By visual inspection, estimates from the model using the 'forecast revision dataset' (blue error bar with $\diamond$) are closer to the SPF data in general.

For the other alternatives, there is no clear visible pattern. Estimates using the 'forecast error dataset' (green error bar with $\triangle$) yield eKG for output close to the estimates with macroeconomic data only. They imply an intermediate degree of information frictions consumption and investment and nominal rates when comparing with revision data and macro data only. Interestingly, it also provides a closer match to the SPF data estimates for inflation than any other dataset.

Finally, estimates using the 'forecast dataset' (red error bar with $\square$) produce reasonable estimates for output and inflation and imply very strong (weak) informational frictions in consumption (nominal rate) data.

If the eKGs are good measures for information frictions, our results suggest that revision data is more informative about informational frictions than data on forecasts or forecast errors.

include any measurement error in forecast revisions or errors by definition.

5.3 Do empirical measures of informational frictions really capture information frictions?

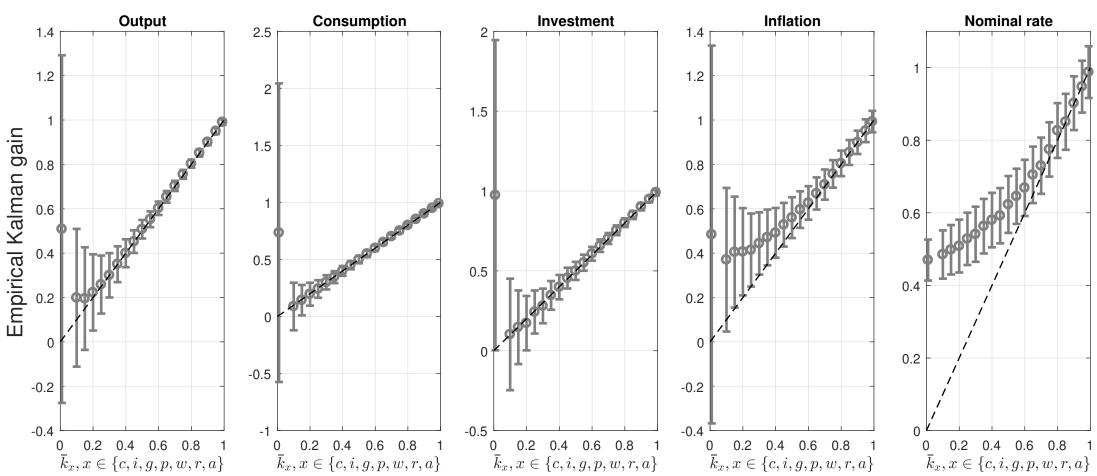
Until now, we used the eKGs as benchmark measure for information frictions to discipline our model estimates. Now, conditional on our model being a good description of the expectational data, we can shed some light on how informative eKGs are regarding actual information frictions.

We pursue three additional experiments. In the baseline, we simulate samples for posterior mean for the 'Forecast revision' dataset. In this section, for each parameter we change its value within a grid and keep other parameters fixed at posterior mean. For each set of new parameters, we simulate $n_{sim} = 100$ samples of time-series $t = 1, \cdots, T_{sim}$ with $T_{sim} = 105$ and get the eKGs for each parameter vector in the grid.

First, we do a sanity check exercise. We keep all parameter fixed at the posterior mean and set all sKGs ($\bar{k}_x$, for $x \in \{c, i, g, p, w, r, a\}$) to the same value, varying from 0 (No info) to 1 (Full info).

Figure 4 shows that estimates of eKGs are closely aligns into the 45^o degree (black dashed) line for output, consumption and investment. Thus, when all information frictions change together in the same direction, eKGs of those variables change in the same proportion.

Figure 4: Empirical Kalman gains as a function of informational frictions



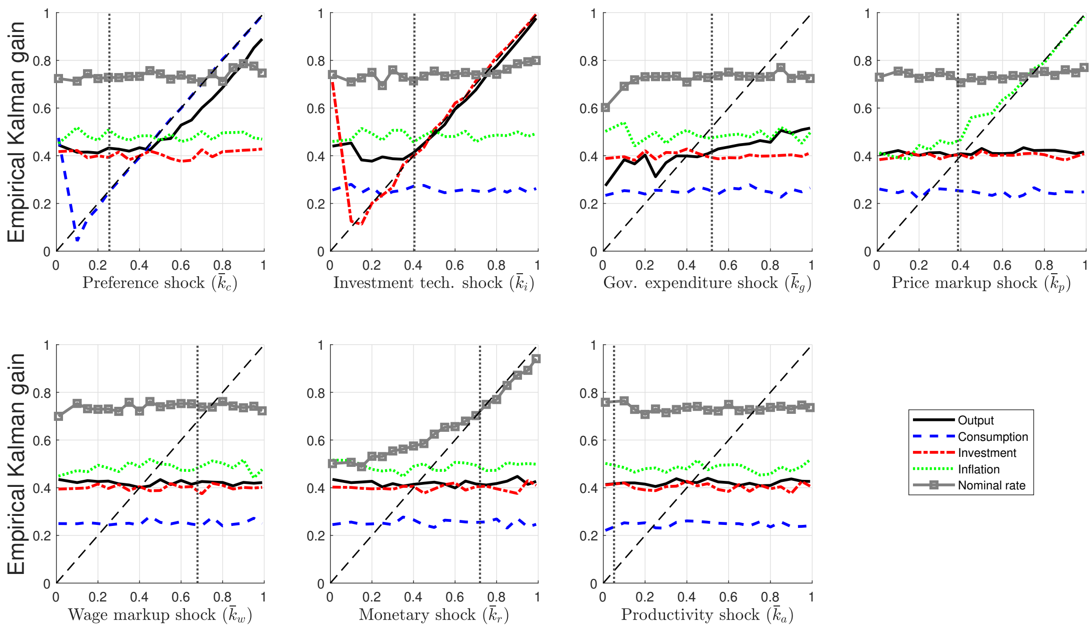
Note: Output, consumption and investment refers to their growth rates. Inflation and interest rates are in levels. The black dashed line represents the 45^o degree line.

The same is not true for inflation and more strongly for the nominal rate. Two patterns stand-out. First, for very strong informational frictions, eKGs tend to *underestimate* the degree of informational frictions for those variables. Second, as sKGs increase, eKGs gets close to the structural values.

If anything, this exercise suggests that informational frictions may be even stronger than suggested by CG's empirical strategy.

Second exercise, instead simulating data by changing all sKGs simultaneously, we do a new simulation by changing each parameter. Figure 5 shows the results of the second experiment.

Figure 5: Empirical Kalman gains as a function of informational frictions



Note: Output, consumption and investment refers to their growth rates. Inflation and interest rates are in levels. The black dotted vertical line represents the posterior mean of each parameter. The black dashed line represents the 45^{o} degree line.

Each eKG is increasing on at least one of the sKGs, which reassures their relevance as measures of those frictions. Information frictions for each shock does not affect all eKGs homogeneously. Interestingly, each eKG is affected differently by each sKG and their relationship are intuitive.

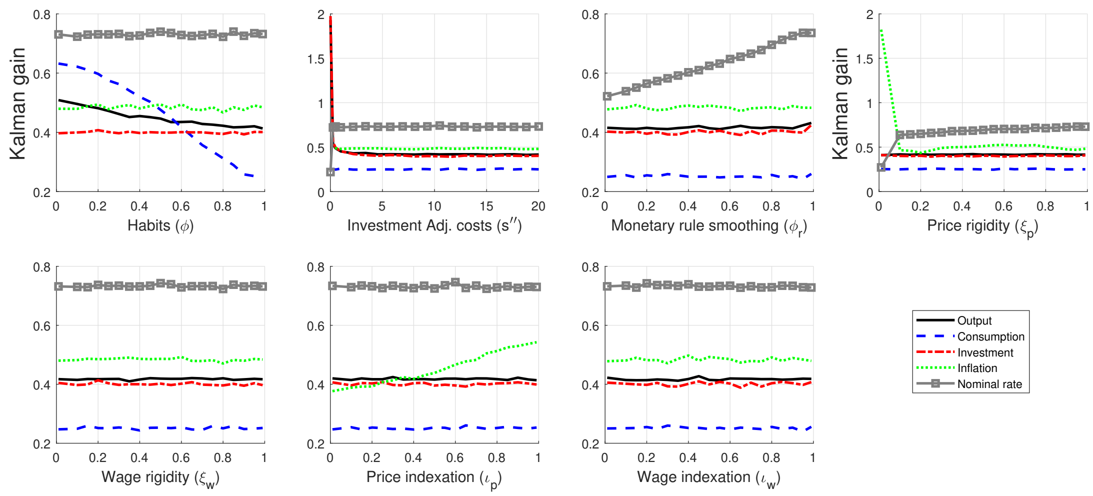
For instance, the information friction on the preference shock ($\bar{k}_c$) plays a major role in the consumption and output eKGs only. Similarly, the monetary shock KG ($\bar{k}_r$) plays a major role in the nominal rate eKG.

Wage markup and productivity information frictions does not play an important role for the selected variables.

In our final exercise, we evaluate how sensitive the eKGs are to changes to seven key parameters of the model that generates endogenous persistence: Habits (φ), investment adjustment costs (s''), monetary rule smoothing (ϕ_r), price rigidity (ξ_p) and indexation (ι_p), wage rigidity (ξ_w) and indexation (ι_w).

Figure 6 shows the results from those simulations. The key finding is that most of the eKGs are remarkably stable when changing the parameter values. There are few exceptions that worth mentioning.

Figure 6: Empirical Kalman gains as a function of real and nominal frictions



Note: Output, consumption and investment refers to their growth rates. Inflation and interest rates are in levels. The black dotted vertical line represents the posterior mean of each parameter.

First, some eKGs are sensitive are specially sensitive to frictions that determine the persistence for those variables for the whole parameter support. Specifically, consumption eKG is sensitive to habits (φ), inflation eKG to price indexation (ι_p), nominal rate eKG is sensitive to monetary rule smoothing (ϕ_r). Price rigidity (ξ_p) also plays a important role for inflation and nominal rate eKGs.

Second, some eKGs are very when some parameters are on extreme values. For instance, when prices are flexible ($\xi_p \to 0$), eKGs for inflation and nominal rates deviate substantially from (in different directions). Output, investment and nominal rate eKGs are also quite sensitive when there are no investment adjustment costs. Since those two are very unlikely scenarios given the literature (and ours) estimates of those two parameters, those results reinforce the robustness of the approach of measuring informational frictions from regressions in the spirit of CG.

However, some variables might be less informative about information frictions than others. For instance, consumption and inflation are more sensitive to real and nominal frictions.

6 Frictions, shocks and business cycles when accounting for informational frictions

This section discusses how the incorporation of informational friction changes our understanding of i) which frictions are most important to fit the data, ii) what drives business cycle fluctuations, and iii) how shocks propagate in the economy.

Can information frictions replace standard frictions?

We find that both standard and informational frictions are essential to fit the macroeconomic and expectation data. Indeed, estimates of real and nominal rigidities increase. This result goes against a strand of the literature that argues, in more theoretical settings, that incomplete information can substitute for the more "ad-hoc" forms of sluggish adjustment usually embedded in macroeconomic models.

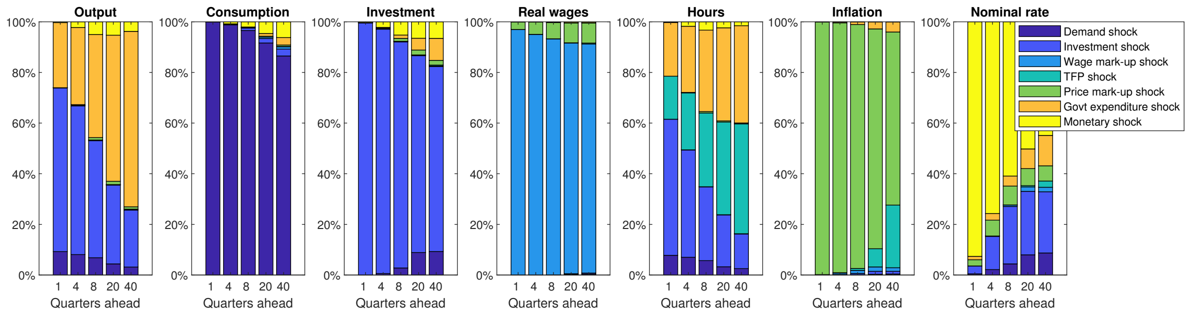
What are the main drivers of business cycles fluctuations?

We perform a forecast error variance decomposition (FEVD) exercise to explore the drivers of business cycles. Figure 7 shows the results of this exercise.

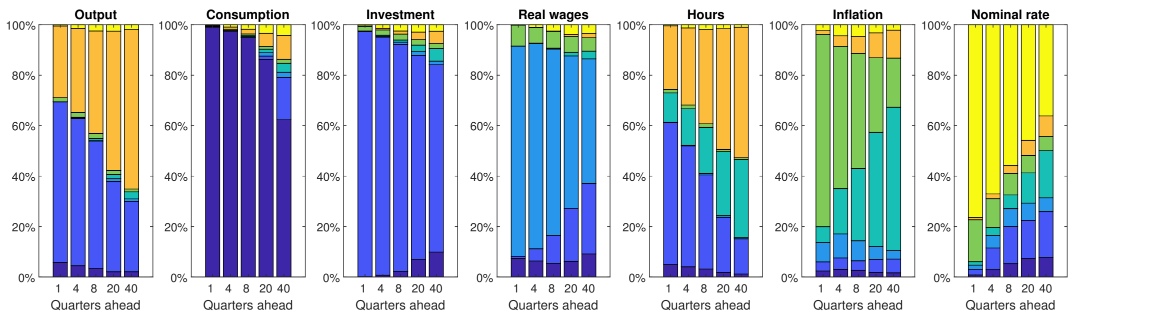
When incorporating data on expectations in the estimation of the DI model, several noteworthy observations arise. First, the variance of consumption forecast errors is predominantly attributed to preference shocks. Second, there is a notable increase in the relevance of government expenditure shocks for output, associated with a decrease in their relevance for consumption. Lastly, the role of monetary shocks in driving fluctuations in the nominal interest rate decreases over longer horizons as other shocks, especially the investment-specific technology shock, become more important.

When using macroeconomic data only, several notable distinctions emerge from the findings of Smets and Wouters (2007) (SW). First, monetary shocks exhibit a relatively higher impact on the variance decompositions of real variables, particularly evident for output and investment. This phenomenon is attributed to the unexpectedly high persistence of monetary shocks in our estimation. This discrepancy relative to SW's findings may be partly explained by differences in their sample period, which spans from 1966:1 to 2004:4, compared to ours, which spans from 1981:1 to 2007:4. Furthermore, our exercise finds that the primary driver of inflation variance is price markup shocks, while SW finds that wage markup shocks also play a significant role in driving inflation fluctuations. Additionally, investment shocks play a more prominent role in explaining output fluctuations in our setting compared to theirs. This observation is consistent with the findings of Justiniano et al. (2011) and Auclert et al. (2020), and can be attributed to the inclusion of inventories in total investment, which SW abstracts from.

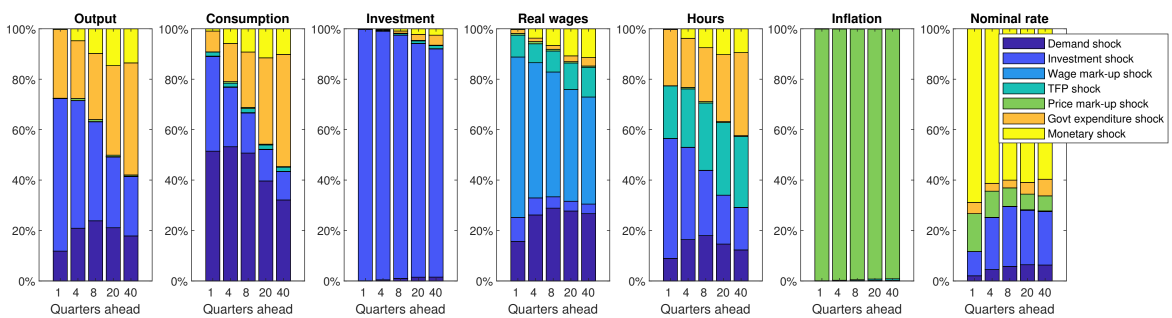
Figure 7: Forecast error variance decomposition for macroeconomic aggregates by model and dataset



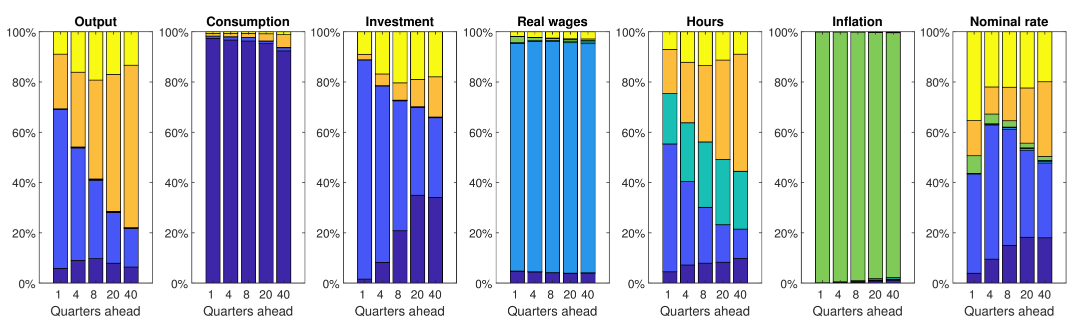
(a) DI model - Forecast revision dataset



(b) FI model - Forecast revision dataset



(c) DI model - Macroeconomic only dataset



(d) FI model - Macroeconomic only dataset

Impulse response functions

In this section, we compare the impulse response functions (IRFs) of the DI and FI models.

To highlight the effect of informational frictions on the dynamic properties of the model, we also compute the IRFs of a FI model with the mean of posterior estimates of the DI model. This comparison helps to disentangle if the DI model dynamics differ from the FI model due to the informational frictions or the estimates from other parameters. Indeed, the differences between the black line and the red line are entirely explained by the informational friction.

The overall similarity in the dynamics of the FI model (blue line) and the DI model (black line) is noticeable, but their impulse responses also feature some key differences. For instance, the investment shock has a more pronounced and persistent effect on real wages, while the opposite holds for the effect of the shock on consumption (Figure 8). Interestingly, these differences are explained entirely by the introduction of dispersed information. Moreover, compared with its full-information counterpart, the DI model generates a weaker recession after inflationary price markup shocks. This is explained by the fact that, when agents do not observe perfectly the price mark-up shock, its impact on price increases is muted, and hence the central bank does not need to raise interest rates and cools the economy down as much as in the FI model (Figure 10). Finally, monetary shocks have stronger effects on macroeconomic aggregates, but a less inflationary impact (Figure 9).

Figure 8: Impulse responses to an investment-specific technological shock

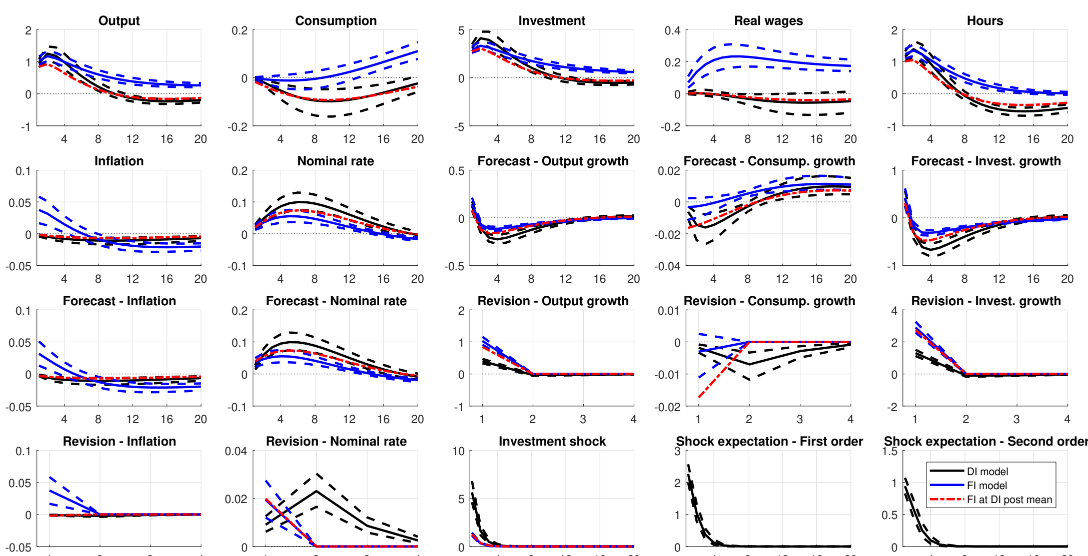


Figure 9: Impulse responses to a monetary shock

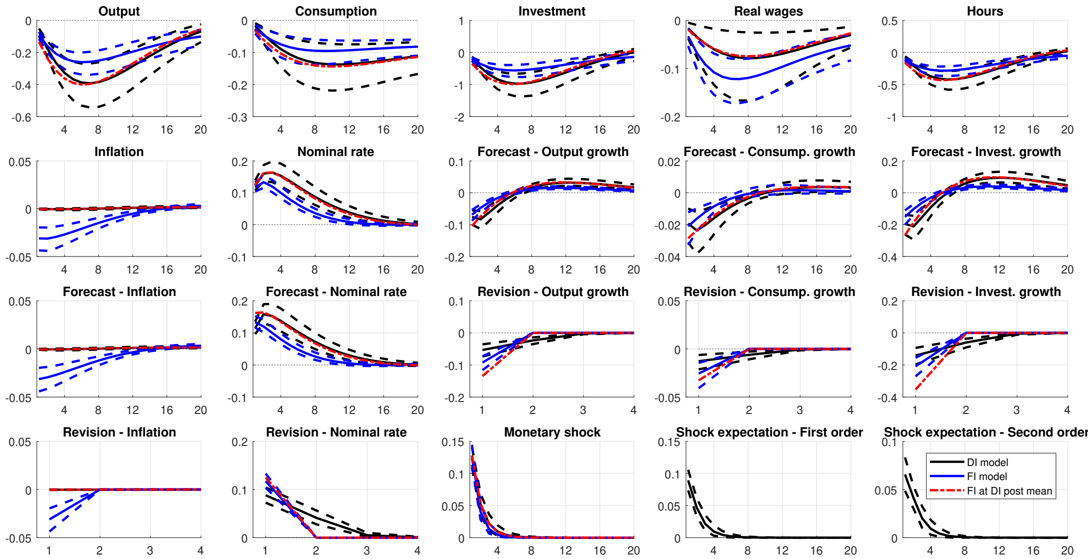
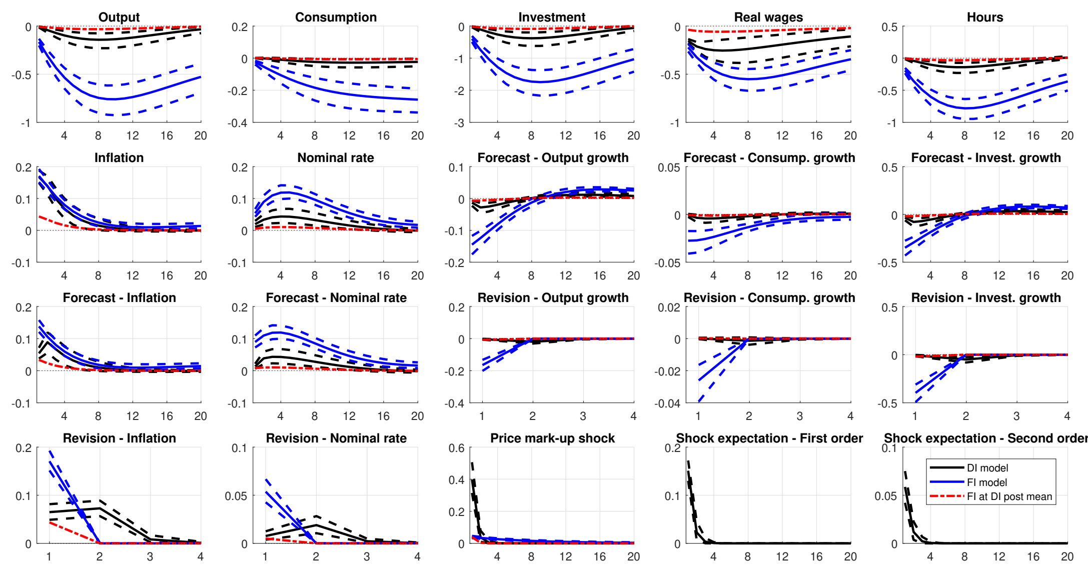


Figure 10: Impulse responses to price markup shock



7 Conclusion

We develop a general solution method that allows enriching a standard medium-scale DSGE model with dispersed information and estimating using Bayesian techniques with comprehensive macroeconomic and expectation data. We draw important conclusions regarding the role information frictions play in business cycles.

First, the degree of informational friction varies significantly across shocks and indicates important departures from complete information. Second, information frictions cannot substitute standard frictions to generate inertia in macroeconomic variables. When disciplining our model with expectational data, standard frictions such as habits and investment costs increase substantially despite the pervasive informational frictions. Third, simulated data from the model can match standard empirical measures of informational frictions when using data on forecast revisions in the estimation. Fourth, we use our model to assess whether those empirical estimates are reliable measures of informational frictions. We find that these measures are more sensitive to information frictions than to standard frictions. Finally, compared with its full-information counterpart, our model generates a weaker recession after inflationary price markup shocks and stronger real effects but a less inflationary response to monetary shocks.

A natural extension for our work is evaluating whether the introduction of a model featuring dispersed information is able to reduce the number of exogenous shocks driving the economy without worsening the fit of the model to the data. Some of the shocks embedded in mainstream DSGE models have debatable microfoundation, hence reducing the dimensionality of the vector of structural shocks would gives us more parsimonious and better microfounded DSGE models to study business cycles. Moreover, embedding a richer informational structure (with public endogenous and exogenous signals, for instance) in our model may improve its performance in explaining business cycle patterns. In conclusion, we see an open venue for exploring the quantitative potential of DSGE models with informational frictions.

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Appendix

A Log-linear model and full derivation

A.1 Optimality conditions

Households

From the households' utility maximization problem, we get the following first-order conditions for consumption and bonds, respectively

$$E_{ht}[\Lambda_{h,t} P_t] = \frac{E_{ht}\left[e^{\eta_t^c}\right]}{C_{h,t} - \varphi C_{h,t-1}} \tag{55}$$

$$E_{ht}[\Lambda_{h,t}] = \beta E_{ht}\left[\Lambda_{h,t+1} R_t\right] \tag{56}$$

where $\Lambda_{h,t}$ is the Lagrange multiplier of households' budget constraint (20). The optimal conditions for capital and investment are given by, respectively:

$$\Phi_{h,t} = \beta E_{ht}\left[\Lambda_{h,t+1}\left(R_{t+1}^k U_{h,t+1} - P_{t+1} a(U_{h,t+1})\right) P_{t+1}\right] + (1-\delta)\beta E_{ht}\left[\Phi_{h,t+1}\right] \tag{57}$$

$$\begin{aligned} E_{ht}[\Lambda_{h,t} P_t] = \Phi_{h,t} E_{ht}\left[e^{\eta_t^i}\right]\left(1 - S\left(\frac{I_{h,t}}{I_{h,t-1}}\right) - S'\left(\frac{I_{h,t}}{I_{h,t-1}}\right)\frac{I_{h,t}}{I_{h,t-1}}\right) \\ +\beta E_{ht}\left[\Phi_{h,t+1} e^{\eta_{t+1}^i} S'\left(\frac{I_{h,t+1}}{I_{h,t}}\right)\left(\frac{I_{h,t+1}}{I_{h,t}}\right)^2\right] \end{aligned} \tag{58}$$

where $\Phi_{h,t}$ is the Lagrange multiplier associated with the law of motion of capital. The optimal value of the capital utilization rate solves

$$E_{ht}[R_t^k] = a'(U_{h,t}) E_{ht}[P_t]. \tag{59}$$

Recall that households can buy state-contingent assets in terms of consumption (but not for leisure). Therefore, it must hold that for all households $h \in [0,1]$ that $C_{h,t} = C_t$, $I_{h,t} = I_t$, $U_{h,t} = U_t$, $\Lambda_{h,t} = \Lambda_t$ and $\Phi_{h,t} = \Phi_t$. However, households still have heterogeneous expectations, $E_{ht}[\cdot]$, wages $w_{h,t}$ and labor supply $L_{h,t}$.

The optimal condition of the wage setting problem is

$$E_{ht}\left[\sum_{s=0}^{\infty}(\beta\xi_w)^s\left(\Lambda_{t,t+s}\frac{X_{t,t+s}^w}{\mu_{t+s}^w} - \frac{1+\mu_{t+s}^w}{\mu_{t+s}^w}\frac{L_{h,t+s}^{\chi}}{W_{h,t}^*}\right)L_{h,t+s}\right] = 0 \tag{60}$$

where $\Lambda_{t,t+s} \equiv \frac{\Lambda_{t+s}}{\Lambda_t}$ is the stochastic discount factor from period t to period $t+s$. The marginal rate of substitution of labor and consumption is defined as

$$MRS_{h,t} \equiv -\frac{U_{L_{h,t}}}{U_{C_{h,t}}} = L_{h,t}^{\chi} \left(C_{h,t} - \varphi C_{h,t-1} \right). \tag{61}$$

Intermediate good firms

At stage 1, firms choose their optimal price based on the information from signals. The optimal price $P_{i,t}^{*}$ that maximize intermediate firm i's profit solve the following first-order condition

$$E_{it} \left[\sum_{s=0}^{\infty} (\beta \xi_p)^s \Lambda_{t,t+s} \left(\frac{X_{t,t+s}}{\mu_{t+s}^{p}} - \frac{1+\mu_{t+s}^{p}}{\mu_{t+s}^{p}} \frac{MC_{i,t+s}}{P_{i,t}^{*}} \right) Y_{i,t+s} \right] = 0. \tag{62}$$

At stage 2, they hire labor at the nominal wage W_t and rent capital at the rental rate R_t^k. Cost minimization subject to production function (5) implies that the capital-labor ratio is given by[23]

$$\frac{K_{i,t}}{L_{i,t}} = \frac{\alpha}{1-\alpha} \frac{W_t}{R_t^k}. \tag{63}$$

We obtain that the marginal cost is given by

$$MC_{i,t} = \frac{1}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}} \frac{(R_t^k)^{\alpha} \left(W_t \right)^{1-\alpha}}{a_t}. \tag{64}$$

Since the production function displays constant return of scale, capital-labor ratios and marginal costs are the same across firms in equilibrium.

[23]Since factor prices are revealed at stage 2, firms does not need to form expectations for W_t and R_t^k.

A.2 Log-linearized model

The log-linearization of optimal conditions leads to the following system of equations

$$c_t = \frac{\varphi/\gamma}{1+\varphi/\gamma} c_{t-1} + \frac{1}{1+\varphi/\gamma} \bar{E}_t[c_{t+1}] - \frac{1-\varphi/\gamma}{1+\varphi/\gamma} \bar{E}_t[r_t - \pi_{t+1} - (x^c_{t+1} - x^c_t)] \tag{65.1}$$

$$i_t = \left(\frac{1}{1+\beta}\right) i_{t-1} + \left(\frac{\beta}{1+\beta}\right) \bar{E}_t[i_{t+1}] + \left(\frac{1}{s''(1+\beta)\gamma^2}\right) \left(q_t + \bar{E}_t\left[x^i_t\right]\right) \tag{65.2}$$

$$q_t = \left(\frac{\beta(1-\delta)}{\gamma}\right) \bar{E}_t[q_{t+1}] + \left(1 - \frac{\beta(1-\delta)}{\gamma}\right) \bar{E}_t\left[r^k_{t+1}\right] - \bar{E}_t[r_t - \pi_{t+1}] \tag{65.3}$$

$$k_t = \left(\frac{1-\delta}{\gamma}\right) k_{t-1} + \left(1 - \frac{1-\delta}{\gamma}\right) \left(i_t + x^i_t\right) \tag{65.4}$$

$$u_t = \left(\bar{R}^k/a''\right) \bar{E}_t[r^k_t] \tag{65.5}$$

$$k^u_t = k_{t-1} + u_t \tag{65.6}$$

$$mrs_t = \left(\frac{1}{1-\varphi/\gamma}\right)(c_t - \varphi/\gamma c_{t-1}) + \chi l_t \tag{65.7}$$

$$y_t = \left(1 + \frac{\Phi_p}{\bar{Y}}\right)(\alpha k^u_t + (1-\alpha) l_t + x^a_t) \tag{65.8}$$

$$k^u_t - l_t = w_t - r^k_t \tag{65.9}$$

$$mc_t = (1-\alpha) w_t + \alpha r^k_t - x^a_t \tag{65.10}$$

$$y_t = \frac{1}{1-g_y}\left(\frac{\bar{C}}{\bar{Y}} c_t + \frac{\bar{I}}{\bar{Y}} i_t + \frac{\bar{R}^k \bar{K}}{\bar{Y}} u_t + x^g_t\right) \tag{65.11}$$

$$r_t = \phi_r r_{t-1} + (1-\phi_r)(\phi_\pi \pi_t + \phi_y y_t) + x^r_t \tag{65.12}$$

$$\begin{aligned} \pi_t =& \xi_p \iota_p \pi_{t-1} + \kappa_p \xi_p \bar{E}_t\left[mc_t + x^p_t\right] + (1-\xi_p)(1-\iota_p \beta \xi_p) \bar{E}_t\left[\pi_t\right] \\ &+ (1-\xi_p)\beta\xi_p \int_0^1 E_{it}[p^*_{i,t+1}] di \end{aligned} \tag{65.13}$$

$$w_t + \pi_t = \xi_w(\iota_w \pi_{t-1} + w_{t-1}) + (1-\xi_w) w^*_t \tag{65.14}$$

$$w^*_t = \frac{(1-\beta\xi_w)}{1+\chi\theta_w} \bar{E}_t\left[mrs_t + w_t + x^w_t\right] + (1-\iota_w\beta\xi_w)\bar{E}_t\left[\pi_t\right] + \beta\xi_p \int_0^1 E_{ht}\left[w^*_{h,t+1}\right] dh \tag{65.15}$$

where $\kappa_p = \frac{(1-\xi_p)(1-\beta\xi_p)}{\xi_p}$, $\kappa_w = \frac{(1-\xi_w)(1-\beta\xi_w)}{\xi_w(1+\chi\theta_w)}$ and $\theta_w = \frac{1+\bar{\mu}^w}{\bar{\mu}^w}$. $\bar{E}_t[\cdot] \equiv \int_0^1 E_{it}[\cdot] di$ denotes average expectation, where $E_{it}[\cdot] \equiv E[\cdot \mid \mathcal{I}_{it}]$ is agent i's expectation based on her information set at period t, $\mathcal{I}_{it}$.

All equations are standard in macroeconomic models. The first equation is the Euler equation for consumption, which is derived by combining the log-linearized versions of equations (55) and (56). The second equation is the equilibrium condition for investment from (58) and the third determines the marginal real value of a unit of capital q_t that come from condition (57). The forth is log-linearized version of the usual law of motion of capital (17). The fifth is the log-linearized version of the optimal condition for capital utilization (59) and the sixth comes from the definition of utilized capital (19). The seventh and eighth are log-linearized versions of the aggregation of the marginal rate of substitution between consumption and labor from equation (61) and the production function (5), respectively. The ninth and tenth comes from aggregating firms' cost minimization condition (63) and marginal cost (63), respectively. The eleventh derives from the aggregate resource constraint (29), while twelfth is the log-linearized version of the Taylor rule (25). Finally, the thirteenth is the New Keynesian Phillips curve for prices that combines log-linearized versions of equations (62) and (10). The last equation is the New Keynesian Phillips curve for wages that combines log-linearized versions of equations (60) and (24).

The derivation of the model can be found in the Online Appendix.

B Data

Data on macroeconomic aggregates are collected from the Federal Reserve Database (FRED) and expectation data come from the Survey of Professional Forecasters (SPF). The time series collected to construct the database are displayed in Table 2. It starts in 4Q81, the first quarters with data for all time series on expectations that I use in this work, and ends in 4Q07, before the Great Recession.

Table 2: Database

Series	**Description**	**Measure**	**Source**
GDPC1	Real Gross Domestic Product	Billions of Chained 2012 Dollars	BEA
PCECC96	Real Personal Consumption Expenditures	Billions of Chained 2012 Dollars	BEA
GPDIC1	Real Gross Private Domestic Investment	Billions of Chained 2012 Dollars	BEA
GDPDEF	Gross Domestic Product: Implicit Price Deflator	Index 2012=100	BEA
FF	Effective Federal Funds Rate	Percentage points	FED Board
PRS85006063	Nonfarm Business Sector: Compensation	Index 2012=100	BEA
HOANBS	Nonfarm Business Sector: Hours of All Persons	Index 2012=100	BLS
CNP16OV	Civilian Noninstitutional Population	Thousands of People	BLS
RGDP	Forecast for the real GDP	Chained Dollars (base year varies)	SPF
RCONSUM	Forecast for the real PCE	Chained Dollars (base year varies)	SPF
RNRESINV	Forecast for the nonresidential fixed investment	Chained Dollars (base year varies)	SPF
PGDP	Forecast for the GDP price index	Index (base year varies)	SPF
TBILL	Forecast for the annual three-month Treasury bill rate	Percentage points	SPF

Note: BEA, BLS, Fed Board and SPF stand for, respectively: U.S. Bureau of Economic Analysis, U.S. Bureau of Labor Statistics, Board of Governors of the Federal Reserve and Survey of Professional Forecasters.

Real output (GDPC1), consumption (PCE) and investment (GDPIC1) are computed as per capita aggregates by dividing them to the Hodrick-Prescott filtered population index (CNP16OV). Real wage is constructed by dividing the compensation in the nonfarm business sector (PRS85006063) by the GDP deflator (GDPDEF). Per capita total hours is computed by dividing the hours of all persons out nonfarm business sector (HOANBS) from the smoothed population index. Inflation is defined as the annualized quarterly growth rate of GDP deflator (GDPDEF). The nominal interest rate is computed by the log of the gross federal funds rate (FF) to be consistent with the log-linearization procedure.

$$\begin{aligned}
Y_t^{obs} &= GDPC1_t/HP(CNP16OV_t) \\
C_t^{obs} &= PCE_t/HP(CNP16OV_t) \\
I_t^{obs} &= GPDIC1_t/HP(CNP16OV_t) \\
W_t^{obs} &= PRS85006063_t/GDPDEF_t \\
dy_t^{obs} &= 100\log(Y_t^{obs}/Y_{t-1}^{obs}) \\
dc_t^{obs} &= 100\log(C_t^{obs}/C_{t-1}^{obs}) \\
di_t^{obs} &= 100\log(I_t^{obs}/I_{t-1}^{obs}) \\
dw_t^{obs} &= 100\log(W_t^{obs}/W_{t-1}^{obs}) \\
L_t^{obs} &= HOANBS_t/HP(CNP16OV_t) \\
\Pi_t^{obs} &= 400\log(GDPDEF_t/GDPDEF_{t-1}) \\
R_t^{obs} &= 100\log(1+FF_t/100)
\end{aligned}$$

Expectation data are constructed as follows

$$
\begin{aligned}
\bar{E}_t[Y_{t+1}^{obs}] &= RGDP3_t/HP(CNP16OV_{t+1}) \\
\bar{E}_t[Y_t^{obs}] &= RGDP2_t/HP(CNP16OV_t) \\
\bar{E}_t[C_{t+1}^{obs}] &= RCONSUM3_t/HP(CNP16OV_{t+1}) \\
\bar{E}_t[C_t^{obs}] &= RCONSUM2_t/HP(CNP16OV_t) \\
\bar{E}_t[I_{t+1}^{obs}] &= RINVEST3_t/HP(CNP16OV_{t+1}) \\
\bar{E}_t[I_t^{obs}] &= RINVEST32_t/HP(CNP16OV_t) \\
\bar{E}_t[dy_t^{obs}] &= 100\log\left(\bar{E}_t[Y_{t+1}^{obs}]/\bar{E}_t[Y_t^{obs}]\right) \\
\bar{E}_t[dy_t^{obs}] &= 100\log\left(\bar{E}_t[Y_t^{obs}]/\bar{E}_t[Y_{t-1}^{obs}]\right) \\
\bar{E}_t[dc_t^{obs}] &= 100\log\left(\bar{E}_t[C_t^{obs}]/\bar{E}_t[C_{t-1}^{obs}]\right) \\
\bar{E}_t[di_t^{obs}] &= 100\log\left(\bar{E}_t[I_t^{obs}]/\bar{E}_t[I_{t-1}^{obs}]\right) \\
\bar{E}_t[\pi_t^{obs}] &= 400\log(PGDP3t/PGDP2t) \\
\bar{E}_t[R_t^{obs}] &= \log(1+TBILL2_t/100) \\
\bar{F}[dy_t] &= dy_t^{obs} - \bar{E}_{t-1}[dy_t^{obs}] \\
\bar{F}[dc_t] &= dc_t^{obs} - \bar{E}_{t-1}[dc_t^{obs}] \\
\bar{F}[di_t] &= di_t^{obs} - \bar{E}_{t-1}[di_t^{obs}] \\
\bar{F}[\pi_t] &= \pi_t^{obs} - \bar{E}_{t-1}[\pi_t^{obs}] \\
\bar{F}[R_t] &= R_t^{obs} - \bar{E}_{t-1}[R_t^{obs}]
\end{aligned}
$$

C Alternative datasets, measurement equations and posteriors

The measurement equations from macroeconomic data are the same as the baseline specification (53). When considering data on forecasts, we consider the following measurement equations

$$
\begin{aligned}
f_{dy,t}^{obs} &= \bar{E}_t[\Delta y_t] + \varepsilon_{dy,t}^{f,me} \\
f_{dc,t}^{obs} &= \bar{E}_t[\Delta c_t] + \varepsilon_{dc,t}^{f,me} \\
f_{di,t}^{obs} &= \bar{E}_t[\Delta i_t] + \varepsilon_{di,t}^{f,me} \\
f_{\pi,t}^{obs} &= 4\bar{E}_t[\pi_t] + \varepsilon_{\pi,t}^{f,me} \\
f_{r,t}^{obs} &= 4\bar{E}_t[r_t] + \varepsilon_{r,t}^{f,me}.
\end{aligned}
$$

This measurement equation is from the specification defined as 'Forecast dataset', which includes macroeconomic and forecast data.

When using data on forecast errors, we use measurement equations given by

$$
\begin{aligned}
e^{obs}_{dy,t} &= \Delta y_t - \bar{E}_{t-1}[\Delta y_t] + \varepsilon^{fe,me}_{dy,t} \\
e^{obs}_{dc,t} &= \Delta c_t - \bar{E}_{t-1}[\Delta c_t] + \varepsilon^{fe,me}_{dc,t} \\
e^{obs}_{di,t} &= \Delta i_t - \bar{E}_{t-1}[\Delta i_t] + \varepsilon^{fe,me}_{di,t} \\
e^{obs}_{\pi,t} &= 4\left(\pi_t - \bar{E}_{t-1}[\pi_t]\right) + \varepsilon^{fe,me}_{\pi,t} \\
e^{obs}_{r,t} &= 4\left(r_t - \bar{E}_{t-1}[r_t]\right) + \varepsilon^{fe,me}_{r,t},
\end{aligned}
$$

This measurement equation is from the specification defined as 'Forecast error dataset', which includes macroeconomic and forecast error data.

Finally, the specification defined as 'Macroeconomic only dataset' uses the first seven equations from (53).

D Posterior estimates from alternative datasets

E Simulated forecast errors and revisions

In section 5.1, we describe the simulation of forecast errors and revisions for the DI model estimated for the 'Forecast revision dataset'. The simulation for estimates for other datasets is analogous, but has to take into account that they differ in their measurement errors. Here we provide a description of those differences.

Forecast error dataset. Again, consider the vector of parameters, Θ, that collects all parameters of the model and the vector of some selected endogenous variables, z_t. For a given Θ, we simulate a n_{sim} samples of those variables and their expectations $\{z_t, \bar{E}_t[z_t], \bar{E}_{t-1}[z_t]\}$ and measurement errors $\{\varepsilon^{me}_{z,t}\}$ for $t = 1, \cdots, T_{sim}$ and $z \in \{dy, dc, di, \pi, r\}$. Then, we construct n_{sim} time-series for average forecast revisions and forecast errors such that

$$
\begin{aligned}
\bar{Rev}^{sim}_{z,t|t-1} &= \bar{E}_t[z_t] - \bar{E}_{t-1}[z_t] \\
\bar{Fe}^{sim}_{z,t-1} &= z_t - \bar{E}_{t-1}[z_t] + \varepsilon^{me,fe}_{z,t}
\end{aligned}
$$

for $t = 1, \cdots, T_{sim}$ and $z \in \{dy, dc, di, \pi, r\}$.

Note that the measurement error is now on the forecast error instead of the forecast revision as in the section 5.1.

Table 3: Posterior distributions for common parameters in alternative datasets

Dataset:	Macroeconomic only			Forecast revision			Forecast error			Forecast		
Posterior:	Mean	5%	95%	Mean	5%	95%	Mean	5%	95%	Mean	5%	95%
Endogenous propagation parameters												
χ	1.82	1.18	2.74	1.84	0.99	3.04	2.12	1.61	2.76	1.04	0.97	1.12
φ	0.33	0.13	0.54	0.98	0.96	0.99	0.98	0.97	0.99	0.95	0.94	0.96
a''	0.31	0.17	0.50	0.46	0.29	0.67	0.43	0.26	0.65	0.66	0.46	0.89
s''	5.16	2.52	8.52	12.35	8.78	16.45	16.27	12.47	20.62	11.42	8.54	14.71
ϕ_y	0.20	0.12	0.29	0.31	0.21	0.42	0.07	0.04	0.12	0.05	0.02	0.08
ϕ_π	1.00	0.60	1.48	1.55	0.99	2.18	3.25	2.63	3.99	3.43	2.86	4.13
ϕ_r	0.71	0.60	0.82	0.92	0.89	0.95	0.89	0.86	0.92	0.80	0.76	0.84
ι_p	0.24	0.04	0.70	0.69	0.48	0.80	0.04	0.01	0.09	0.04	0.01	0.08
ι_w	0.49	0.16	0.82	0.39	0.12	0.72	0.49	0.17	0.81	0.47	0.15	0.80
ξ_p	0.98	0.97	1.00	0.98	0.98	0.99	0.88	0.84	0.92	0.69	0.67	0.71
ξ_w	0.08	0.02	0.17	0.92	0.85	0.96	0.80	0.74	0.85	0.10	0.03	0.18
Steady-state parameters												
π_{ss}	2.29	1.82	2.71	1.49	1.06	1.96	2.74	2.29	3.20	2.26	1.78	2.76
r_{ss}	7.40	6.41	8.43	7.44	6.38	8.57	6.35	5.56	7.21	5.50	4.68	6.34
γ_{ss}	0.40	0.35	0.45	0.47	0.45	0.50	0.34	0.32	0.37	0.22	0.20	0.25
Structural shocks parameters												
ρ_c	0.89	0.79	0.96	0.13	0.07	0.20	0.24	0.16	0.32	0.36	0.27	0.44
ρ_i	0.48	0.35	0.61	0.29	0.22	0.35	0.16	0.10	0.21	0.12	0.09	0.15
ρ_g	0.99	0.97	1.00	0.99	0.98	1.00	0.99	0.98	1.00	1.00	1.00	1.00
ρ_p	0.54	0.12	0.76	0.10	0.03	0.21	0.83	0.76	0.88	0.91	0.89	0.93
ρ_w	0.89	0.80	0.96	0.34	0.21	0.48	0.24	0.14	0.34	0.64	0.56	0.72
ρ_r	0.91	0.80	0.98	0.41	0.29	0.54	0.53	0.43	0.63	0.98	0.96	0.99
ρ_a	0.96	0.91	0.99	0.998	0.995	0.999	0.90	0.85	0.94	0.96	0.95	0.98
σ_c	0.13	0.08	0.22	1.72	1.28	2.29	0.32	0.21	0.52	11.98	6.91	18.30
σ_i	1.72	1.21	2.35	5.65	4.63	6.84	2.09	1.47	3.12	3.31	2.47	4.40
σ_g	0.53	0.45	0.62	0.50	0.43	0.58	0.57	0.50	0.65	0.57	0.50	0.66
σ_p	0.11	0.05	0.21	0.39	0.31	0.50	0.09	0.07	0.11	0.12	0.10	0.14
σ_w	0.62	0.48	0.84	0.91	0.55	1.50	1.54	0.85	2.79	1.36	0.86	2.17
σ_r	0.12	0.11	0.13	0.12	0.11	0.14	0.13	0.12	0.15	0.16	0.14	0.18
σ_a	0.42	0.37	0.47	0.44	0.39	0.49	0.41	0.36	0.46	0.42	0.37	0.47
Information friction parameters												
$\bar{k}_c$	0.68	0.54	0.82	0.25	0.20	0.31	0.80	0.60	0.95	0.05	0.03	0.08
$\bar{k}_i$	0.84	0.71	0.95	0.40	0.36	0.45	0.81	0.62	0.94	0.61	0.49	0.72
$\bar{k}_g$	0.60	0.48	0.73	0.52	0.49	0.55	0.81	0.71	0.91	0.73	0.70	0.76
$\bar{k}_p$	0.82	0.62	0.96	0.39	0.30	0.47	0.58	0.47	0.70	0.87	0.77	0.95
$\bar{k}_w$	0.79	0.61	0.93	0.68	0.41	0.90	0.46	0.23	0.69	0.38	0.23	0.52
$\bar{k}_r$	0.20	0.07	0.36	0.72	0.63	0.80	0.68	0.59	0.77	0.14	0.12	0.17
$\bar{k}_a$	0.90	0.78	0.97	0.05	0.04	0.07	0.59	0.41	0.80	0.55	0.50	0.61

Forecast dataset. For the 'Forecast dataset', we first construct the forecasts with their own measurement errors and compute the implied forecast error and revision. The forecast is

Table 4: Prior and Posterior distributions for measurement errors in alternative datasets

Parameters	Prior			Posterior		
	Dist.	Mean	Std. Dev.	Mean	5%	95%
Forecast revision dataset						
σ_{dy}^{me}	$\mathcal{IG}$	0.1	0.02	0.29	0.26	0.33
σ_{dc}^{me}	$\mathcal{IG}$	0.1	0.02	0.21	0.18	0.23
σ_{di}^{me}	$\mathcal{IG}$	0.5	0.2	1.22	1.09	1.37
σ_{π}^{me}	$\mathcal{IG}$	0.15	0.05	0.45	0.40	0.50
σ_{r}^{me}	$\mathcal{IG}$	0.15	0.05	0.25	0.22	0.29
Forecast error dataset						
$\sigma_{dy}^{me,fe}$	$\mathcal{IG}$	0.15	0.05	0.45	0.40	0.51
$\sigma_{dc}^{me,fe}$	$\mathcal{IG}$	0.15	0.05	0.40	0.36	0.45
$\sigma_{di}^{me,fe}$	$\mathcal{IG}$	1	0.25	2.19	1.96	2.44
$\sigma_{\pi}^{me,fe}$	$\mathcal{IG}$	0.2	0.05	1.12	1.01	1.25
$\sigma_{r}^{me,fe}$	$\mathcal{IG}$	0.2	0.05	0.23	0.20	0.26
Forecast dataset						
$\sigma_{dy}^{me,f}$	$\mathcal{IG}$	0.1	0.02	0.20	0.18	0.23
$\sigma_{dc}^{me,f}$	$\mathcal{IG}$	0.1	0.02	0.18	0.16	0.20
$\sigma_{di}^{me,f}$	$\mathcal{IG}$	0.3	0.05	0.79	0.71	0.88
$\sigma_{\pi}^{me,f}$	$\mathcal{IG}$	0.3	0.05	0.46	0.40	0.53
$\sigma_{r}^{me,f}$	$\mathcal{IG}$	0.7	0.25	0.57	0.49	0.66

constructed as

$$\bar{E}_{z,t|t-1}^{sim} = \bar{E}_{t-1}[z_t] + \varepsilon_{z,t}^{me,f}$$

for $t = 1, \cdots, T_{sim}$ and $z \in \{dy, dc, di, \pi, r\}$.

$$\bar{Rev}_{z,t|t-1}^{sim} = \bar{E}_t[z_t] - \bar{E}_{z,t|t-1}^{sim} = \bar{E}_t[z_t] - \bar{E}_{t-1}[z_t] - \varepsilon_{z,t}^{me,f}$$
$$\bar{Fe}_{z,t-1}^{sim} = z_t - \bar{E}_{z,t|t-1}^{sim} = z_t - \bar{E}_{t-1}[z_t] - \varepsilon_{z,t}^{me,f}$$

Note that the measurement error for the one-step ahead forecast, $\varepsilon_{z,t}^{me,f}$, shows up in both forecast errors and revisions.

For other specifications, measurement errors introduces noise, which does not affect the estimated β from regression 54 but increases its standard deviation. Here, the measurement error induces a one-to-one relationship that potentially bias the estimate of β. The size of the bias is likely to depend on the relative importance of the measurement error explaining the forecasts.

The posterior means for measurement error standard deviations are roughly two times higher than the prior for all forecasts except for nominal rates (which is less than the prior). This implies that the measurement error explains a small fraction of total variation of the forecast data (roughly 5% for nominal rates and 10% for others). Simulations without the measurement error leads to similar point estimates.

F Solution method and Algorithm

We first start with two Lemmas. The first writes the first-order expectation of the hierarchy as a linear transformation of the hierarchy.

Lemma 1. *The first-order expectation of the hierarchy of expectations satisfy*

$$E_t^{(1)}\left[x_t^{(0:\bar{k})}\right] = Tx_t^{(0:\bar{k})} \tag{66}$$

where $T = \begin{bmatrix} 0_{n\bar{k}\times n} & I_{n\bar{k}} \\ 0_{n\times n} & 0_{n\times n\bar{k}} \end{bmatrix}$ *is the order transformation matrix.*

Proof. By the definition of $x_t^{(0:\bar{k})}$ one can see that $E_t^{(1)}\left[x_t^{(0:\bar{k})}\right] = x_t^{(1:\bar{k}+1)}$. By definition for $\bar{k}$, any order s such that $s > \bar{k}$ does not affect the equilibrium. Then, without loss of generality, one can set $E_t^{(s)}[x_t] = 0$ if $s > \bar{k}$. Therefore, one can rewrite $E_t^{(1)}\left[x_t^{(0:\bar{k})}\right]$ as

$$E_t^{(1)}\left[x_t^{(0:\bar{k})}\right] = \begin{bmatrix} x_t^{(1:\bar{k})} \\ E_t^{(\bar{k}+1)}[x_t] \end{bmatrix} = \begin{bmatrix} x_t^{(1:\bar{k})} \\ 0_{n\times 1} \end{bmatrix} = \begin{bmatrix} 0_{n\bar{k}\times n} & I_{n\bar{k}} \\ 0_{n\times n} & 0_{n\times n\bar{k}} \end{bmatrix} \begin{bmatrix} x_t \\ x_t^{(1:\bar{k})} \end{bmatrix} = Tx_t^{(0:\bar{k})}, \tag{67}$$

where the first equality uses the definition of $x_t^{(1:\bar{k}+1)}$ and the second equality uses that $E_t^{(\bar{k}+1)}[x_t] = 0$. In the last equality, T is defined accordingly.

□

This is Lemma builds on Proposition 1 from Online Appendix of Melosi (2017). It explores the truncation of the hierarchy.

The second Lemma does the opposite: writes the hierarchy as a linear transformation of the first-order expectation of the hierarchy.

Lemma 2. $x_t^{(0:\bar{k})}$ *can be rewritten as a linear function of* x_t *and its average expectation such that*

$$x_t^{(0:\bar{k})} \equiv e_x' x_t + T' E_t^{(1)}\left[x_t^{(0:\bar{k})}\right],$$

Proof. By definition, $x_t^{(0:\bar{k})}$ can be decomposed as $x_t^{(0:\bar{k})} = \left[x_t' \quad (x_t^{(\bar{1}:\bar{k})})'\right]'$. Using this decomposition, the average expectation decomposed as $E_t^{(1)}\left[x_t^{(0:\bar{k})}\right] = \left[(x_t^{(\bar{1}:\bar{k})})' \quad E_t^{(\bar{k}+1)}\left[x_t\right]'\right]'$. Therefore, one can use this two definitions such that

$$x_t^{(0:\bar{k})} \equiv \begin{bmatrix} x_t \\ x_t^{(\bar{1}:\bar{k})} \end{bmatrix} = \begin{bmatrix} I_n \\ 0_{n\bar{k}\times n} \end{bmatrix} x_t + \begin{bmatrix} 0_{n\times n\bar{k}} & 0_{n\times n} \\ I_{n\bar{k}} & 0_{n\bar{k}\times n} \end{bmatrix} \begin{bmatrix} x_t^{(1:\bar{k})} \\ E_t^{(\bar{k}+1)}\left[x_t\right] \end{bmatrix} = e_x' x_t + T' E_t^{(1)}\left[x_t^{(0:\bar{k})}\right],$$

where last equality uses the definitions of T in Lemma 1 and e_x. □

Lemmas are taken from Ribeiro (2018, chap. 3).

F.1 Proof of Proposition 1

Individual and average expectations about the hierarchy. The Kalman filter delivers the individual conditional expectation, $E_{it}\left[\cdot\right] \equiv E\left[\cdot|\mathcal{I}_t^i\right]$, where $\mathcal{I}_t^i = \{s_{i,\tau}, \tau \leq t\}$ is the information set of individual i in period t.

The state equation is the hierarchy of expectations that is given by the state equation (42), restated for convenience:

$$x_t^{(0:\bar{k})} = \mathbf{A} x_{t-1}^{(0:\bar{k})} + \mathbf{B}\varepsilon_t, \tag{68}$$

and the agent i with the observational equation (39), also restated:

$$s_{i,t} = C_x x_t + D v_{i,t}.$$

Note that $s_{i,t}$ is a signal about the shocks and not the whole hierarchy of expectations.

Let the selection matrix $e_x \equiv \begin{bmatrix} I_n & 0_{n\times n\bar{k}} \end{bmatrix}$ such that $x_t = e_x x_t^{(0:\bar{k})}$. Then, we can rewrite the signal in terms of the hierarchy as

$$s_{i,t} = C x_t^{(0:\bar{k})} + D v_{it}. \tag{69}$$

where $C = C_x e_x$.

Each agent i uses the Kalman filter and find the update equation given by

$$E_{i,t}\left[x_t^{(0:\bar{k})}\right] = E_{i,t-1}\left[x_t^{(0:\bar{k})}\right] + \mathbf{K_t}\left[s_{i,t} - E_{i,t-1}\left[s_{i,t}\right]\right], \tag{70}$$

where $\mathbf{K_t}$ is the Kalman gain given by

$$\mathbf{K_t} = \mathbf{P_{t/t-1}} C' \left[C \mathbf{P_{t/t-1}} C' + D\Sigma_v D'\right]^{-1}. \tag{71}$$

As usual, the mean squared error (MSE) of the one-period ahead prediction error is given by

$$\mathbf{P_{t+1/t}} = \mathbf{A}\left[\mathbf{P_{t/t-1}} - \mathbf{K_t}\left[C\mathbf{P_{t/t-1}}\right]\right]\mathbf{A}' + \mathbf{B}\Sigma_\varepsilon\mathbf{B}'. \tag{72}$$

For details of this deviation, see for instance Hamilton (1995, chap. 13).

Using the observational equation, (69), taking expectations and inserting in (70) one can find:

$$E_{it}\left[x_t^{(0:\bar{k})}\right] = E_{i,t-1}\left[x_t^{(0:\bar{k})}\right] + \mathbf{K_t}\left[Cx_t^{(0:\bar{k})} + Dv_{it} - CE_{i,t-1}\left[x_t^{(0:\bar{k})}\right]\right] \tag{73}$$

Therefore, one can rewrite the equation above as

$$E_{it}\left[x_t^{(0:\bar{k})}\right] = \left(I_k - \mathbf{K_t}C\right)E_{it-1}\left[x_t^{(0:\bar{k})}\right] + \mathbf{K_t}\left[Cx_t^{(0:\bar{k})} + Dv_{it}\right] \tag{74}$$

where $k = n(\bar{k}+1)$. Using the fact that $E_{i,t-1}\left[x_t^{(0:\bar{k})}\right] = \mathbf{A}E_{i,t-1}\left[x_{t-1}^{(0:\bar{k})}\right]$ and substituting equation (68), one can find:

$$E_{it}\left[x_t^{(0:\bar{k})}\right] = \left(I_k - \mathbf{K_t}C\right)\mathbf{A}E_{i,t-1}\left[x_{t-1}^{(0:\bar{k})}\right] + \mathbf{K_t}C\mathbf{A}x_{t-1}^{(0:\bar{k})} + \mathbf{K_t}C\mathbf{B}\varepsilon_t + \mathbf{K_t}Dv_{it} \tag{75}$$

We follow the literature by focusing in the stationary equilibrium. Therefore, the expectation of each individual i in the stationary equilibrium is the one which the MSE is in steady-state, i.e., agents update their forecast based on the steady-state Kalman gain. In other words, the dynamics of expectations depends only in the properties of the process they are forecasting and signals, but do not depend in the period t.

Combining equations (71-72), one can find the Riccatti equation

$$\mathbf{P_{t+1/t}} = \mathbf{A}\left[\mathbf{P_{t/t-1}} - \mathbf{P_{t/t-1}}C'\left[C\mathbf{P_{t/t-1}}C' + D\Sigma_v D'\right]^{-1}C\mathbf{P_{t/t-1}}\right]\mathbf{A}' + \mathbf{B}\Sigma_\varepsilon\mathbf{B}' \tag{76}$$

Therefore, one need to iterate this equation to find the steady-state MSE, $\bar{\mathbf{P}}$, and compute its counterpart Kalman gain, $\bar{\mathbf{K}}$. Nimark (2017) shows that if is x_t stationary process, then the expectations hierarchy about this process, $x_t^{(0:\bar{k})}$, is also stationary. This and the fact that Σ_ε is positive definite, then there exists a steady-state solution such that $\bar{\mathbf{P}} = \mathbf{P_{t+1|t}} = \mathbf{P_{t|t-1}}$ which implies the steady-state Kalman gain $\bar{\mathbf{K}} = \mathbf{K_t} = \mathbf{K_{t-1}}$ (see Hamilton; 1995, chap. 13).

Expressions (44) in the Proposition 1 are equations (71-72) using the steady-state Kalman gain, $\bar{\mathbf{K}}$, instead of $\bar{\mathbf{K}}_\mathbf{t}$.

Moreover, the average expectation is easily computed by

$$\bar{E}_t\left[x_t^{(0:\bar{k})}\right] \equiv \int_0^1 E_{it}\left[x_t^{(0:\bar{k})}\right]di = \left(I_k - \bar{\mathbf{K}}C\right)\mathbf{A}\bar{E}_{t-1}\left[x_{t-1}^{(0:\bar{k})}\right] + \bar{\mathbf{K}}C\mathbf{A}x_{t-1}^{(0:\bar{k})} + \bar{\mathbf{K}}C\mathbf{B}\varepsilon_t \tag{77}$$

Analogously, the individual and average expectation of Proposition 1 are equations (75) and (77) using the steady-state Kalman gain, $\bar{\mathbf{K}}$, instead of $\bar{\mathbf{K}}_{\mathbf{t}}$.

Verify guess for $x_t^{(0:\bar{k})}$. In the first part of the proof, we found the average expectation (77) for the guessed the dynamics of the hierarchy of expectations (68).

Now we verify the guessed hierarchy and find the coefficients $(\mathbf{A}, \mathbf{B})$ consistent with the average expectation.

Substituting the average expectation from equation (77) into the expression from Lemma 2 one can find that:

$$x_t^{(0:\bar{k})} = e_x' x_t + T' \left[\left(I_k - \bar{\mathbf{K}} C \right) \mathbf{A} E_{t-1}^{(1)} \left[x_{t-1}^{(0:\bar{k})} \right] + \bar{\mathbf{K}} C \mathbf{A} x_{t-1}^{(0:\bar{k})} + \bar{\mathbf{K}} C \mathbf{B} \varepsilon_t \right]$$

Then, using the shocks definition (38) and the fact that $x_t = e_x x_t^{(0:\bar{k})}$ one can rewrite equation above as

$$x_t^{(0:\bar{k})} = e_x' \left(A_1 e_x x_{t-1}^{(0:\bar{k})} + \varepsilon_t \right) + T' \left(I_k - \bar{\mathbf{K}} C \right) A E_{t-1}^{(1)} \left[x_{t-1}^{(0:\bar{k})} \right] + T' \bar{\mathbf{K}} C A x_{t-1}^{(0:\bar{k})} + T' \bar{\mathbf{K}} C \mathbf{B} \varepsilon_t$$

Using Lemma 1 at period $t-1$ into equation above and rearranging:

$$x_t^{(0:\bar{k})} = \left[e_x' A_1 e_x + T' \left(I_k - \bar{\mathbf{K}} C \right) A T + T' \bar{\mathbf{K}} C A \right] x_{t-1}^{(0:\bar{k})} + \left[T' \bar{\mathbf{K}} C \mathbf{B} + e_x' \right] \varepsilon_t. \tag{78}$$

This expression shows that the expectations hierarchy is a function of its lag and structural shocks, as guessed in equation (68). Therefore, the expression above verifies that $x_t^{(0:\bar{k})}$ follows the guessed form and the square brackets terms provide identities for A and $\mathbf{B}$ in equations (43)

□

F.2 Proof of Proposition 2

This proof builds on techniques developed by Ribeiro (2018, chap. 3). The key difference is that we guess a law of motion for individual endogenous variables instead of guessing the aggregate. This allows recovering expressions for solving for $(\mathbf{Q_0}, \mathbf{Q_1})$ and then get the solution for $\mathbf{Q}$. We also consider the exogenous information case only.

The guessed law of motion for individual endogenous variables is given (40), restated for convenience:

$$Y_{i,t} = \mathbf{R} Y_{i,t-1} + \mathbf{Q_0} x_t + \mathbf{Q_1} E_{i,t} \left[x_t^{(0:\bar{k})} \right]. \tag{79}$$

Aggregating the individual law of motion (79):

$$Y_t = \mathbf{R}Y_{t-1} + \mathbf{Q_0}x_t + \mathbf{Q_1}E_t^{(1)}\left[x_t^{(0:\bar{k})}\right] \tag{80}$$

$$= \mathbf{R}Y_{t-1} + \mathbf{Q_0}x_t + \mathbf{Q_1}Tx_t^{(0:\bar{k})} \tag{81}$$

where the last equality uses Lemma 1.

By computing the individual expectation of the endogenous aggregate variables, one can find that

$$\begin{aligned} E_{it}\left[Y_t\right] &= \mathbf{R}Y_{t-1} + \mathbf{Q_0}E_{it}\left[x_t\right] + \mathbf{Q_1}TE_{it}\left[x_t^{(0:\bar{k})}\right] \\ &= \mathbf{R}Y_{t-1} + (\mathbf{Q_0}e_x + \mathbf{Q_1}T)E_{it}\left[x_t^{(0:\bar{k})}\right] \\ E_t^{(1)}\left[Y_t\right] &= \mathbf{R}Y_{t-1} + (\mathbf{Q_0}e_x + \mathbf{Q_1}T)E_t^{(1)}\left[x_t^{(0:\bar{k})}\right] \end{aligned} \tag{82}$$

where the first equality uses that Y_{t-1} is known, second equality uses the definition of the selection matrix, e_x, and the third equality aggregates.

Similarly, the individual expectation for endogenous variables in $t+1$:

$$\begin{aligned} E_{it}\left[Y_{t+1}\right] &= \mathbf{R}E_{it}\left[Y_t\right] + \mathbf{Q_0}E_{it}\left[x_{t+1}\right] + \mathbf{Q_1}TE_{it}\left[x_{t+1}^{(0:\bar{k})}\right] \\ &= \mathbf{R}E_{it}\left[Y_t\right] + (\mathbf{Q_0}A_1e_x + \mathbf{Q_1}T\mathbf{A})E_{it}\left[x_t^{(0:\bar{k})}\right] \\ &= \mathbf{R}^2Y_{t-1} + \left[\mathbf{R}(\mathbf{Q_0}e_x + \mathbf{Q_1}T) + (\mathbf{Q_0}A_1e_x + \mathbf{Q_1}T\mathbf{A})\right]E_{it}\left[x_t^{(0:\bar{k})}\right] \\ E_t^{(1)}\left[Y_{t+1}\right] &= \mathbf{R}^2Y_{t-1} + \left[\mathbf{R}(\mathbf{Q_0}e_x + \mathbf{Q_1}T) + (\mathbf{Q_0}A_1e_x + \mathbf{Q_1}T\mathbf{A})\right]E_t^{(1)}\left[x_t^{(0:\bar{k})}\right] \end{aligned} \tag{83}$$

where the second equality uses equations (38) and (42), and the definition of e_x. Third equation uses equation (82) and the fourth aggregates.

Finally, taking the individual expectation of the law of motion (40) in period $t+1$ implies that

$$\begin{aligned} E_{it}[Y_{i,t+1}] &= \mathbf{R}Y_{i,t} + \mathbf{Q_0}E_{it}[x_{t+1}] + \mathbf{Q_1}E_{it}\left[E_{i,t+1}\left[x_{t+1}^{(0:\bar{k})}\right]\right] \\ &= \mathbf{R}Y_{i,t} + (\mathbf{Q_0}A_1e_x + \mathbf{Q_1}\mathbf{A})E_{it}\left[x_t^{(0:\bar{k})}\right] \end{aligned}$$

where the second equality uses the law of iterated expectations (which holds for i's expectation but not for the average expectation), equations (38) and (42) as before.

Aggregating the expectation above leads to

$$\begin{aligned}
\int_0^1 E_{it}[Y_{i,t+1}]di &= \mathbf{R}Y_t + (\mathbf{Q_0}A_1e_x + \mathbf{Q_1}\mathbf{A})E_t^{(1)}[x_t^{(0:\bar{k})}] \\
&= \mathbf{R}\left[\mathbf{R}Y_{t-1} + \mathbf{Q_0}x_t + \mathbf{Q_1}E_t^{(1)}\left[x_t^{(0:\bar{k})}\right]\right] + (\mathbf{Q_0}A_1e_x + \mathbf{Q_1}\mathbf{A})E_t^{(1)}[x_t^{(0:\bar{k})}] \\
&= \mathbf{R}^2Y_{t-1} + \mathbf{R}\mathbf{Q_0}x_t + [\mathbf{R}\mathbf{Q_1} + (\mathbf{Q_0}A_1e_x + \mathbf{Q_1}\mathbf{A})]\,E_t^{(1)}[x_t^{(0:\bar{k})}]
\end{aligned} \tag{84}$$

where the second equality uses equation (81). Note that $\int_0^1 E_{it}[Y_{i,t+1}]di \neq E_t^{(1)}[Y_{t+1}]$.

Substituting the guessed solution (81) and the expectations (82-84) into the system of equations (37) one can find:

$$\begin{aligned}
&F_1\left[\mathbf{R}^2Y_{t-1} + \mathbf{R}\mathbf{Q_0}x_t + [\mathbf{R}\mathbf{Q_1} + (\mathbf{Q_0}A_1e_x + \mathbf{Q_1}\mathbf{A})]\,E_t^{(1)}[x_t^{(0:\bar{k})}]\right] + \\
&F_2\left[\mathbf{R}^2Y_{t-1} + [\mathbf{R}(\mathbf{Q_0}e_x + \mathbf{Q_1}T) + (\mathbf{Q_0}A_1e_x + \mathbf{Q_1}T\mathbf{A})]\,E_t^{(1)}\left[x_t^{(0:\bar{k})}\right]\right] + \\
&G_1\left[\mathbf{R}Y_{t-1} + \mathbf{Q_0}x_t + \mathbf{Q_1}E_t^{(1)}\left[x_t^{(0:\bar{k})}\right]\right] + G_2\left[\mathbf{R}Y_{t-1} + (\mathbf{Q_0}e_x + \mathbf{Q_1}T)E_t^{(1)}\left[x_t^{(0:\bar{k})}\right]\right] + \\
&HY_{t-1} + [(L_1A_1 + L_2)]\,e_xE_t^{(1)}\left[x_t^{(0:\bar{k})}\right] + M_1x_t = 0_{m\times 1}
\end{aligned}$$

which can be rearranged to

$$\begin{aligned}
&\left[F\mathbf{R}^2 + G\mathbf{R} + H\right]Y_{t-1} + [(F_1\mathbf{R} + G_1)\,\mathbf{Q_0} + M_1]\,x_t \\
&[F_1\,[\mathbf{R}\mathbf{Q_1} + (\mathbf{Q_0}A_1e_x + \mathbf{Q_1}\mathbf{A})] + F_2\,[\mathbf{R}(\mathbf{Q_0}e_x + \mathbf{Q_1}T) + (\mathbf{Q_0}A_1e_x + \mathbf{Q_1}T\mathbf{A})] + \\
&G_1\mathbf{Q_1} + G_2(\mathbf{Q_0}e_x + \mathbf{Q_1}T) + (L_1A_1 + L_2 + M)e_x]\,E_t^{(1)}\left[x_t^{(0:\bar{k})}\right] = 0_{m\times 1}
\end{aligned}$$

which can be simplified to

$$\begin{aligned}
&\left[F\mathbf{R}^2 + G\mathbf{R} + H\right]Y_{t-1} + [[F_1\mathbf{R} + G_1]\,\mathbf{Q_0} + M_1]\,x_t + \\
&[[F_1\mathbf{R} + G_1]\,\mathbf{Q_1} + F_1\mathbf{Q_1}\mathbf{A} + (F_2\mathbf{R} + G_2)\mathbf{Q_1}T + F_2\mathbf{Q_1}T\mathbf{A} \\
&[(F_2\mathbf{R} + G_2)\mathbf{Q_0} + (F_1 + F_2)\mathbf{Q_0}A_1 + (L_1A_1 + L_2)]\,e_x]\,E_t^{(1)}\left[x_t^{(0:\bar{k})}\right] = 0_{m\times 1}
\end{aligned}$$

This condition must hold for all realizations of Y_{t-1}, x_t and $E_t^{(1)}\left[x_t^{(0:\bar{k})}\right]$. Therefore, all coefficients between square brackets must be zero, which leads to

$$
\begin{aligned}
&F\mathbf{R}^2 + G\mathbf{R} + H = 0_{m\times m}\\
&\left[F_1\mathbf{R} + G_1\right]\mathbf{Q_0} + M_1 = 0_{m\times n}\\
&\left[F_1\mathbf{R} + G_1\right]\mathbf{Q_1} + F_1\mathbf{Q_1}\mathbf{A} + (F_2\mathbf{R} + G_2)\mathbf{Q_1}T + F_2\mathbf{Q_1}T\mathbf{A}\\
&\left[(F_2\mathbf{R} + G_2)\mathbf{Q_0} + F\mathbf{Q_0}A_1 + (LA_1 + M_2)\right]e_x = 0_{m\times k}
\end{aligned}
$$

which leads to the same equations from Proposition 2. $\mathbf{R}$ can be solved using Uhlig (2001) method. For a given solution of $\mathbf{R}$, $\mathbf{Q_0}$ and $\mathbf{Q_1}$ can be solved by straightforward vectorization as discussed in the Proposition.

□

F.3 Algorithm

Algorithm. Set the initial values $(\mathbf{A}^{(0)}, \mathbf{B}^{(0)})$, a small tolerance $\epsilon > 0$ and set $i = 1$. Then, follow the steps:

1. Given $\mathbf{A} = \mathbf{A}^{(i-1)}$ and $\mathbf{B}^{(i-1)}$, compute $\bar{\mathbf{K}}$ and $\bar{\mathbf{P}}$ using equations (44) using standard solver for Ricatti equations. Set $\bar{\mathbf{K}}^{(i)} = \bar{\mathbf{K}}$ and $\bar{\mathbf{P}}^{(i)} = \bar{\mathbf{P}}$.

2. Given $\mathbf{A}^{(i-1)}$ and $\bar{\mathbf{K}}^{(i)}$, compute the right hand side of equations (43) and solve for $\mathbf{B}$ and $\mathbf{A}$ the equations by matrix inversion. Set $\mathbf{B}^{(i)} = \mathbf{B}$, $\mathbf{A}^{(i)} = \mathbf{A}$.

3. If $\max\left\{||\mathbf{B}^{(i)} - \mathbf{B}^{(i-1)}||, ||\mathbf{A}^{(i)} - \mathbf{A}^{(i-1)}||, ||\mathbf{P}^{(i)} - \mathbf{P}^{(i-1)}||\right\} < \epsilon$, stop iterating. Otherwise, set $i = i + 1$ and go back to step 1.

Given the solution for $\mathbf{A}$, one can use standard techniques for solving for $(\mathbf{R}, \mathbf{Q_0}, \mathbf{Q_1}, \mathbf{Q})$ using Proposition 2.

G Impulse response functions

In this section, the remaining impulse responses for the FI and ICK models estimated with the dataset including expectation data are displayed.

Figure 11: Impulse responses to preference shock

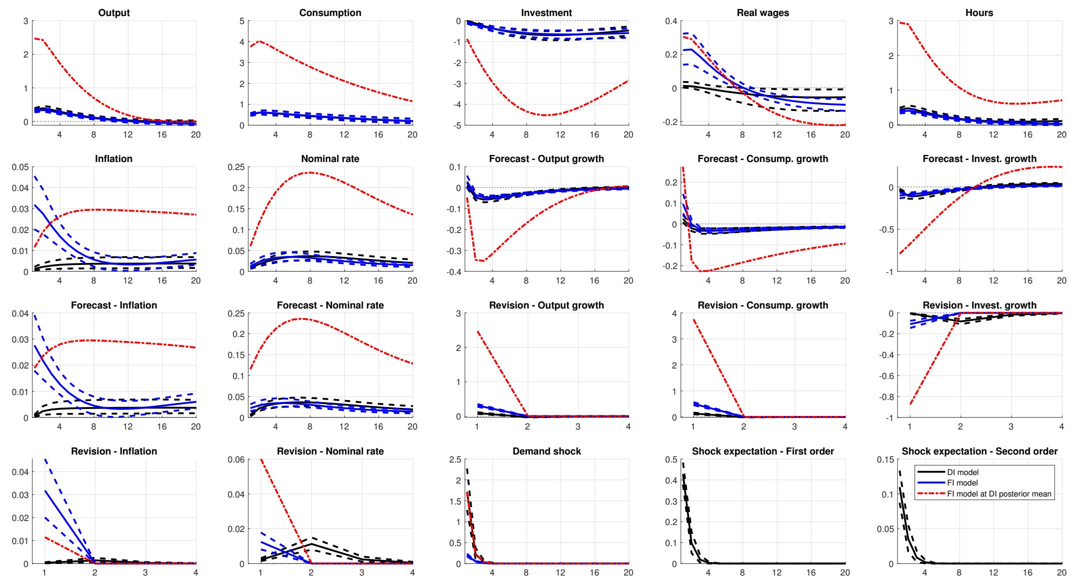

Figure 12: Impulse responses to TFP shock

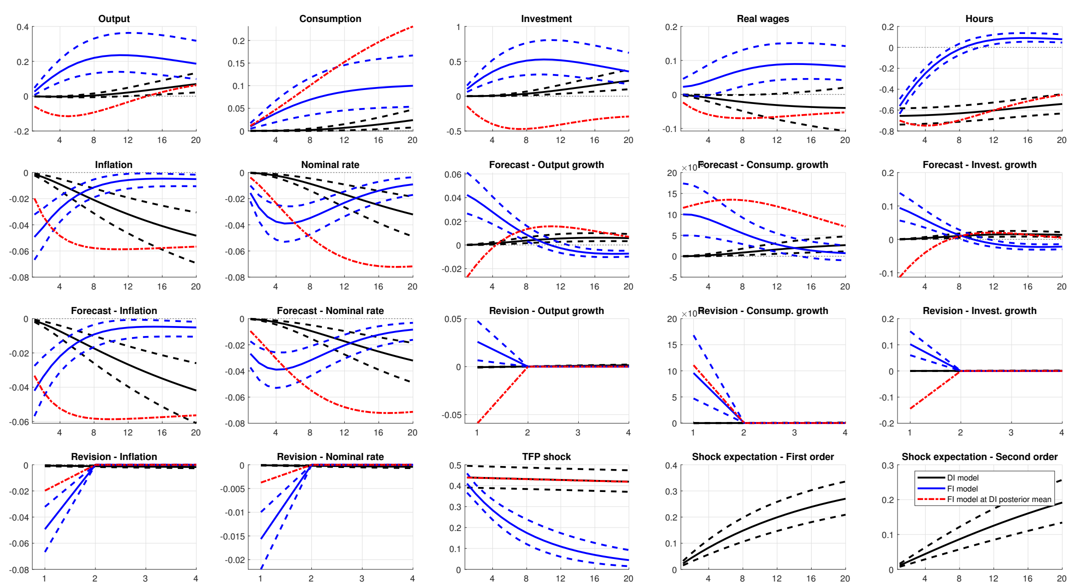

Figure 13: Impulse responses to wage mark-up shock

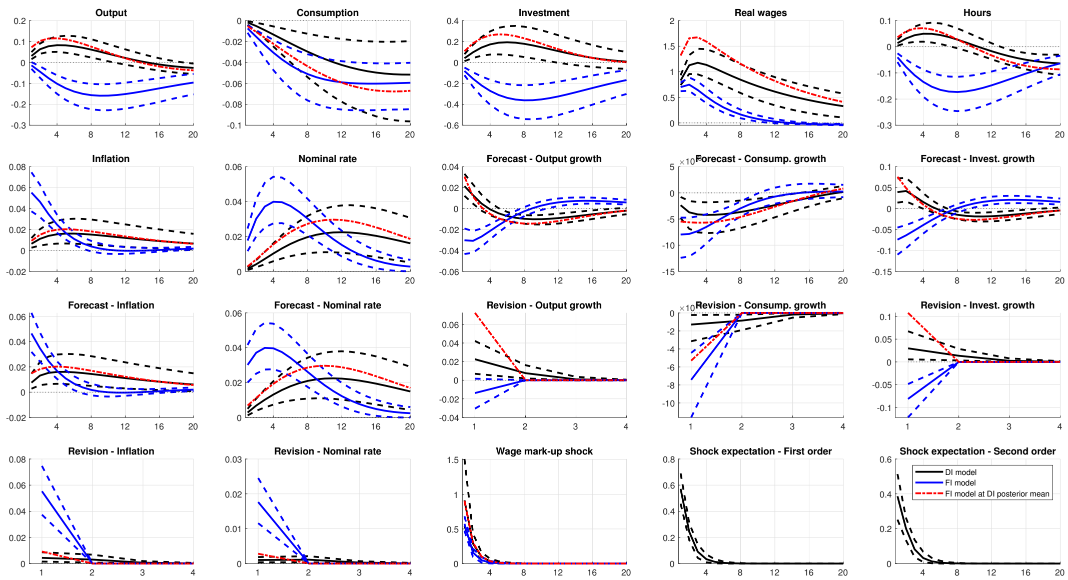

Figure 14: Impulse responses to price mark-up shock

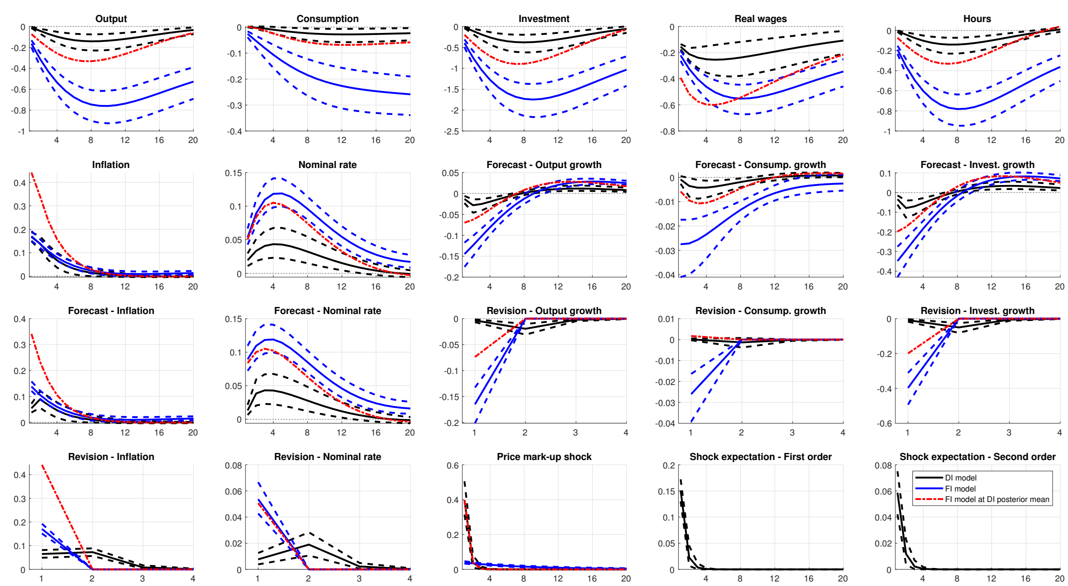

Figure 15: Impulse responses to government expenditure shock